The stratified spaces of a symplectic Lie group action

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Abstract

In this paper we show that the classical symplectic stratification theorem [17] of the reduced spaces of a canonical group action in a symplectic manifold can be obtained even in the absence of a momentum map by replacing this object by its natural generalization, the cylinder valued momentum map introduced by Condevaux, Dazord, and Molino [3]. In the process of proving this result we will provide a normal form for the cylinder valued momentum map.

Keywords: symplectic reduction, momentum maps, symplectic normal form, symplectic Lie group action, symplectic stratification theorem.

1 Introduction

Let (M, ω) be a connected paracompact symplectic manifold acted upon properly and canonically by a Lie group G. Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{g}^*$ its dual. Assume for the moment that the action admits a standard equivariant momentum map $\mathbf{J} : M \rightarrow \mathfrak{g}^*$. The Marsden-Weinstein reduction theorem [10] states that if the G-action on M is free then the quotient spaces $M_\mu := \mathbf{J}^{-1}(\mu)/G_\mu$, $\mu \in \mathfrak{g}^*$ inherit a natural symplectic structure from M. The symbol G_μ denotes the isotropy subgroup of $\mu \in \mathfrak{g}^*$ with respect to the coadjoint action. If the G-action is not free, the reduced spaces M_μ are, in general, not regular quotient manifolds. However, if one thinks of M_μ as a quotient topological space, Sjamaar and Lerman [17] have proved that it admits a natural stratification whose pieces are symplectic and that this decomposition of the space has the smoothness and local triviality properties that make it into a ***cone space*** (see [15] for the definition of this concept).

Our goal in this paper is to obtain a similar result for *any* symplectic action, even when a momentum map does not exist. Our approach is based on a construction due to Condevaux, Dazord, and Molino [3] that naturally generalizes the standard momentum map to a ***cylinder valued momentum map*** $\mathbf{K} : M \rightarrow \mathfrak{g}^*/\overline{\mathcal{H}}$, that *always* exists for *any* symplectic Lie group action. The quotient $\mathfrak{g}^*/\overline{\mathcal{H}}$ is topologically a cylinder $\mathbb{R}^a \times \mathbb{T}^b$, $a, b \in \mathbb{N}$, and it is obtained as the orbit space of an action of the closure $\overline{\mathcal{H}}$ of $\mathcal{H}$. The group $\mathcal{H}$ is a zero-dimensional Lie subgroup of $(\mathfrak{g}^*, +)$ and is the holonomy of a flat connection on the trivial principal fiber bundle $\pi : M \times \mathfrak{g}^* \rightarrow M$ with $(\mathfrak{g}^*, +)$ as Abelian structure group. This flat connection is constructed using exclusively the canonical G-action and the symplectic form ω on M thereby justifying the name ***Hamiltonian holonomy*** for $\mathcal{H}$.

The reduction by a free action in this context has been studied in [14]. At this level of generality, reduction turns out to be a highly non-trivial procedure. For instance, the Marsden-Weinstein reduced space in this setup is not symplectic but Poisson and three different reduced spaces naturally arise depending on the reduction point of view that one adopts, namely, foliation reduction, Marsden-Weinstein

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reduction, or optimal reduction (see [14] for details). These three spaces coincide in the presence of a standard momentum map (in which case $\mathcal{H} = \{0\}$) or, more generally, when the Hamiltonian holonomy is closed in $\mathfrak{g}^*$, that is, when $\overline{\mathcal{H}} = \mathcal{H}$. In the generalization of the Sjamaar-Lerman stratification theorem to this setup we will always assume that the zero-dimensional submanifold $\mathcal{H}$ is closed in $\mathfrak{g}^*$. Otherwise, it is not clear which of the three reduced spaces found in [14] should exhibit a symplectic stratification as in [17] or if this result is even true in this very general context. The hypothesis $\overline{\mathcal{H}} = \mathcal{H}$ is satisfied for very important examples; for instance, we will show that it holds in the presence of a ***Lie group valued momentum map*** [11, 1].

The main result. *Let (M, ω) be a connected paracompact symplectic manifold and G a Lie group acting properly on it by symplectic diffeomorphisms. Let $\mathbf{K} : M \to \mathfrak{g}^*/\overline{\mathcal{H}}$ be a cylinder valued momentum map for this action. Let $[\mu] \in \mathfrak{g}^*/\overline{\mathcal{H}}$ and let $M_{[\mu]} := \mathbf{K}^{-1}([\mu])/G_{[\mu]}$ be the corresponding quotient space. If $\mathcal{H}$ is a closed subset of $\mathfrak{g}^*$ then the manifolds $M_{[\mu]}^{(H)} := [\mathbf{K}^{-1}([\mu]) \cap G_{[\mu]} M_H^z]/G_{[\mu]}$, with H any isotropy subgroup of the G-action on M and $z \in M_H$, are naturally symplectic and they form a symplectic Whitney stratification of the quotient $M_{[\mu]}$. The quotient $M_{[\mu]}$ is a cone space when considered as a stratified space with strata $M_{[\mu]}^{(H)}$.*

All the elements in this statement will be carefully introduced in the following section. In order to make this result easier to understand we can say at this point that the group $G_{[\mu]}$ is the isotropy subgroup of $[\mu] \in \mathfrak{g}^*/\mathcal{H}$ with respect to a natural G-action on $\mathfrak{g}^*/\mathcal{H}$ that makes $\mathbf{K}$ equivariant. We also recall that the ***isotropy type manifold*** M_H corresponding to the isotropy subgroup H is defined as $M_H := \{m \in M \mid G_m = H\}$; G_m is the isotropy subgroup of the element $m \in M$. Given $z \in M_H$, the submanifold M_H^z is the connected component of M_H that contains z. The set $G_{[\mu]} M_H^z$ denotes its $G_{[\mu]}$ saturation.

As it was the case with the classical stratification theorem of Sjamaar and Lerman, the main tool for the proof of this result is the use of a normal form that provides a convenient expression of the cylinder valued momentum map in a G-invariant neighborhood of any point of M. For the standard momentum map this is the so called Marle-Guillemin-Sternberg normal form [8, 9, 5]. In the cylinder valued momentum map context we will construct this normal form using a symplectic slice theorem whose existence was shown in [16, 12].

Notations and general assumptions. Manifolds: In this paper all manifolds are finite dimensional. **Lie group actions:** The image of a point m in a manifold M under a group action $\Phi : G \times M \to M$ is denoted interchangeably by $\Phi(g, m) = \Phi_g(m) = g \cdot m$, for any $g \in G$. The symbol $L_g : G \to G$ (respectively $R_g : G \to G$) denotes left (respectively right) translation on G by the group element $g \in G$. The group orbit containing $m \in M$ is denoted by $G \cdot m$ and its tangent space by $T_m(G \cdot m)$ or $\mathfrak{g} \cdot m$. The Lie algebra of the group G is usually denoted by $\mathfrak{g}$. Given any $\xi \in \mathfrak{g}$, the symbol ξ_M denotes the infinitesimal generator vector field associated to ξ defined by $\xi_M(m) = \frac{d}{dt}\big|_{t=0} \exp t\xi \cdot m$, for any $m \in M$. **Lie algebra actions:** A ***right (left) Lie algebra action*** of $\mathfrak{g}$ on M is a Lie algebra (anti)homomorphism $\xi \in \mathfrak{g} \longmapsto \xi_M \in \mathfrak{X}(M)$ such that the mapping $(m, \xi) \in M \times \mathfrak{g} \longmapsto \xi_M(m) \in TM$ is smooth. If $\mathfrak{g}$ acts on a symplectic manifold (M, ω) we say that the $\mathfrak{g}$-action is ***canonical*** if $\pounds_{\xi_M} \omega = 0$, for any $\xi \in \mathfrak{g}$. **The Chu map:** Given a symplectic manifold (M, ω) acted canonically upon by a Lie algebra $\mathfrak{g}$, the Chu map $\Psi : M \to Z^2(\mathfrak{g})$, where $Z^2(\mathfrak{g})$ denotes the vector space of real valued Lie algebra two-cocycles on $\mathfrak{g}$, is defined by the expression $\Psi(m)(\xi, \eta) := \omega(m)(\xi_M(m), \eta_M(m))$, for any $m \in M$, $\xi, \eta, \in \mathfrak{g}$. **Annihilators and orthogonal complements:** If $\langle \cdot, \cdot \rangle : W^* \times W \to \mathbb{R}$ is a nondegenerate duality pairing and $V \subset W$, define the ***annihilator*** subspace $V^\circ := \{\alpha \in W^* \mid \langle \alpha, v \rangle = 0 \text{ for all } v \in V\} \subset W^*$ and similarly for a subset of W^*. If (S, ω) is a symplectic vector space and $U \subset S$, define the ω-***orthogonal subspace*** as $U^\omega := \{s \in S \mid \omega(s, u) = 0 \text{ for all } u \in U\}$.

2 The cylinder valued momentum map

In this section we quickly review the cylinder valued momentum map and its elementary properties. This construction, first introduced by Condevaux, Dazord, and Molino in [3] under the name of "reduced momentum map", is the keystone of the main result in this paper. For the proofs of the results stated in the following paragraphs see [14, 13].

The definition of the cylinder valued momentum map. Let (M, ω) be a connected paracompact symplectic manifold and let $\mathfrak{g}$ be a Lie algebra that acts canonically on M. Take the Cartesian product $M \times \mathfrak{g}^*$ and let $\pi : M \times \mathfrak{g}^* \to M$ be the projection onto M. Consider π as the bundle map of the trivial principal fiber bundle $(M \times \mathfrak{g}^*, M, \pi, \mathfrak{g}^*)$ that has $(\mathfrak{g}^*, +)$ as Abelian structure group. The group $(\mathfrak{g}^*, +)$ acts on $M \times \mathfrak{g}^*$ by $\nu \cdot (m, \mu) := (m, \mu - \nu)$, with $m \in M$ and $\mu, \nu \in \mathfrak{g}^*$. Let $\alpha \in \Omega^1(M \times \mathfrak{g}^*; \mathfrak{g}^*)$ be the connection one-form defined by

$$\langle \alpha(m, \mu)(v_m, \nu), \xi \rangle := (\mathbf{i}_{\xi_M} \omega)(m)(v_m) - \langle \nu, \xi \rangle, \tag{2.1}$$

where $(m, \mu) \in M \times \mathfrak{g}^*$, $(v_m, \nu) \in T_m M \times \mathfrak{g}^*$, $\langle \cdot, \cdot \rangle$ denotes the natural pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$, and ξ_M is the infinitesimal generator vector field associated to $\xi \in \mathfrak{g}$. The connection α is flat (see appendix 6.1). For $(z, \mu) \in M \times \mathfrak{g}^*$, let $(M \times \mathfrak{g}^*)(z, \mu)$ be the holonomy bundle through (z, μ) and let $\mathcal{H}(z, \mu)$ be the holonomy group of α with reference point (z, μ) (which is an Abelian zero dimensional Lie subgroup of $\mathfrak{g}^*$ by the flatness of α). The principal bundle $((M \times \mathfrak{g}^*)(z, \mu), M, \pi|_{(M \times \mathfrak{g}^*)(z,\mu)}, \mathcal{H}(z, \mu))$ is a reduction of the principal bundle $(M \times \mathfrak{g}^*, M, \pi, \mathfrak{g}^*)$; it is here that we used the paracompactness of M since it is a technical hypothesis in the Reduction Theorem. To simplify notation, we will write $(\widetilde{M}, M, \widetilde{p}, \mathcal{H})$ instead of $((M \times \mathfrak{g}^*)(z, \mu), M, \pi|_{(M \times \mathfrak{g}^*)(z,\mu)}, \mathcal{H}(z, \mu))$. Let $\widetilde{\mathbf{K}} : \widetilde{M} \subset M \times \mathfrak{g}^* \to \mathfrak{g}^*$ be the projection into the $\mathfrak{g}^*$-factor.

Let $\overline{\mathcal{H}}$ be the closure of $\mathcal{H}$ in $\mathfrak{g}^*$. Since $\overline{\mathcal{H}}$ is a closed subgroup of $(\mathfrak{g}^*, +)$, the quotient $C := \mathfrak{g}^*/\overline{\mathcal{H}}$ is a cylinder (that is, it is isomorphic to the Abelian Lie group $\mathbb{R}^a \times \mathbb{T}^b$ for some $a, b \in \mathbb{N}$). Let $\pi_C : \mathfrak{g}^* \to \mathfrak{g}^*/\overline{\mathcal{H}} = C$ be the projection. Define $\mathbf{K} : M \to C$ to be the map that makes the following diagram commutative:

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{\widetilde{\mathbf{K}}} & \mathfrak{g}^* \\ {\scriptstyle \widetilde{p}}\big\downarrow & & \big\downarrow{\scriptstyle \pi_C} \\ M & \xrightarrow{\mathbf{K}} & \mathfrak{g}^*/\overline{\mathcal{H}}. \end{array} \tag{2.2}$$

In other words, $\mathbf{K}$ is defined by $\mathbf{K}(m) = \pi_C(\nu)$, where $\nu \in \mathfrak{g}^*$ is any element such that $(m, \nu) \in \widetilde{M}$.

We will refer to $\mathbf{K} : M \to \mathfrak{g}^*/\overline{\mathcal{H}} =: C$ as a ***cylinder valued momentum map*** associated to the canonical $\mathfrak{g}$-action on (M, ω) and to $\mathcal{H}$ as the ***Hamiltonian holonomy*** of the $\mathfrak{g}$-action on (M, ω).

Elementary properties. The cylinder valued momentum map is a strict generalization of the standard (Kostant-Souriau) momentum map since the $\mathfrak{g}$-action has a standard momentum map if and only if the holonomy group $\mathcal{H}$ is trivial. In such a case the cylinder valued momentum map *is* a standard momentum map. The cylinder valued momentum map satisfies Noether's Theorem, that is, for any $\mathfrak{g}$-invariant function $h \in C^\infty(M)^{\mathfrak{g}} := \{f \in C^\infty(M) \mid \mathbf{d}h(\xi_M) = 0 \text{ for all } \xi \in \mathfrak{g}\}$, the flow F_t of its associated Hamiltonian vector field X_h satisfies the identity $\mathbf{K} \circ F_t = \mathbf{K}|_{\mathrm{Dom}(F_t)}$. Additionally, for any $v_m \in T_m M$, $m \in M$, $T_m\mathbf{K}(v_m) = T_\mu \pi_C(\nu)$, where $\mu \in \mathfrak{g}^*$ is any element such that $\mathbf{K}(m) = \pi_C(\mu)$ and $\nu \in \mathfrak{g}^*$ is uniquely determined by $\langle \nu, \xi \rangle = (\mathbf{i}_{\xi_M}\omega)(m)(v_m)$, for any $\xi \in \mathfrak{g}$. Also, $\ker(T_m\mathbf{K}) = \left(\left(\mathrm{Lie}(\overline{\mathcal{H}})\right)^\circ \cdot m\right)^\omega$ and $\operatorname{range}(T_m\mathbf{K}) = T_\mu \pi_C\left((\mathfrak{g}_m)^\circ\right)$ (Bifurcation Lemma).

Equivariance properties of the cylinder valued momentum map. Suppose now that the $\mathfrak{g}$-Lie algebra action on (M, ω) is obtained from a canonical action of the Lie group G on (M, ω) by taking the infinitesimal generators of all elements in $\mathfrak{g}$. There is a G-action on the target space of the cylinder valued momentum map $\mathbf{K} : M \to \mathfrak{g}^*/\overline{\mathcal{H}}$ with respect to which it is G-equivariant. This action is constructed by noticing first that the Hamiltonian holonomy $\mathcal{H}$ is invariant under the coadjoint action, that is, $\mathrm{Ad}^*_{g^{-1}}\mathcal{H} \subset \mathcal{H}$, for any $g \in G$. Actually, if G is connected, then $\mathcal{H}$ is pointwise fixed by the coadjoint action. Hence, there is a unique group action $\mathcal{A}d^* : G \times \mathfrak{g}^*/\overline{\mathcal{H}} \to \mathfrak{g}^*/\overline{\mathcal{H}}$ such that for any $g \in G$, $\mathcal{A}d^*_{g^{-1}} \circ \pi_C = \pi_C \circ \mathrm{Ad}^*_{g^{-1}}$. With this in mind, we define $\overline{\sigma} : G \times M \to \mathfrak{g}^*/\overline{\mathcal{H}}$ by

$$\overline{\sigma}(g, m) := \mathbf{K}(\Phi_g(m)) - \mathcal{A}d^*_{g^{-1}}\mathbf{K}(m).$$

Since M is connected by hypothesis, it can be shown that $\overline{\sigma}$ does not depend on the points $m \in M$ and hence it defines a map $\sigma : G \rightarrow \mathfrak{g}^*/\overline{\mathcal{H}}$ which is a group valued one-cocycle: for any $g, h \in G$, it satisfies the equality $\sigma(gh) = \sigma(g) + \mathcal{A}d^*_{g^{-1}}\sigma(h)$. This guarantees that the map

$$\begin{array}{rccc} \Theta : & G \times \mathfrak{g}^*/\overline{\mathcal{H}} & \longrightarrow & \mathfrak{g}^*/\overline{\mathcal{H}} \\ & (g, \pi_C(\mu)) & \longmapsto & \mathcal{A}d^*_{g^{-1}}(\pi_C(\mu)) + \sigma(g), \end{array}$$

defines a G-action on $\mathfrak{g}^*/\overline{\mathcal{H}}$ with respect to which the cylinder valued momentum map $\mathbf{K}$ is G-equivariant, that is, for any $g \in G$, $m \in M$, we have

$$\mathbf{K}(\Phi_g(m)) = \Theta_g(\mathbf{K}(m)).$$

We will refer to $\sigma : G \rightarrow \mathfrak{g}^*/\overline{\mathcal{H}}$ as the ***non-equivariance one-cocycle*** of the cylinder valued momentum map $\mathbf{K} : M \rightarrow \mathfrak{g}^*/\overline{\mathcal{H}}$ and to Θ as the ***affine G-action*** on $\mathfrak{g}^*/\overline{\mathcal{H}}$ induced by σ. The infinitesimal generators of the affine G-action on $\mathfrak{g}^*/\overline{\mathcal{H}}$ are given by the expression

$$\xi_{\mathfrak{g}^*/\overline{\mathcal{H}}}(\pi_C(\mu)) = -T_\mu \pi_C \left(\Psi(m)(\xi, \cdot)\right), \tag{2.3}$$

for any $\xi \in \mathfrak{g}$, $(m, \mu) \in \widetilde{M}$, where $\Psi : M \rightarrow Z^2(\mathfrak{g})$ is the Chu map defined by $\Psi(\xi, \eta) := \omega\left(\xi_M, \eta_M\right)$, for any $\xi, \eta \in \mathfrak{g}$.

The Poisson structures on $\mathfrak{g}^*/\overline{\mathcal{H}}$. The bracket $\{\cdot, \cdot\}_{\mathfrak{g}^*/\overline{\mathcal{H}}} : C^\infty(\mathfrak{g}^*/\overline{\mathcal{H}}) \times C^\infty(\mathfrak{g}^*/\overline{\mathcal{H}}) \rightarrow \mathbb{R}$ defined by

$$\{f, g\}_{\mathfrak{g}^*/\overline{\mathcal{H}}}(\pi_C(\mu)) = \Psi(m)\left(\frac{\delta(f \circ \pi_C)}{\delta \mu}, \frac{\delta(g \circ \pi_C)}{\delta \mu}\right),$$

where $f, g \in C^\infty(\mathfrak{g}^*/\overline{\mathcal{H}})$, $(m, \mu) \in \widetilde{M}$, $\pi_C : \mathfrak{g}^* \rightarrow \mathfrak{g}^*/\overline{\mathcal{H}}$ is the projection, and $\Psi : M \rightarrow Z^2(\mathfrak{g})$ is the Chu map, defines a Poisson structure on $\mathfrak{g}^*/\overline{\mathcal{H}}$ such that $\mathbf{K} : M \rightarrow \mathfrak{g}^*/\overline{\mathcal{H}}$ is a Poisson map.

3 A normal form for the cylinder valued momentum map

The main technical tool in the proof of the classical stratification theorem of Sjamaar and Lerman [17] for the reduced spaces of a proper symplectic action that admits a standard coadjoint equivariant momentum map was a normal form that had been obtained years before by Marle [8, 9] and by Guillemin and Sternberg [5]. This normal form is a version of the classical Slice Theorem for proper group actions adapted to the symplectic symmetric setup that provides a semi-global set of coordinates (global only in the direction of the group orbits) in which the standard momentum map takes a particularly convenient and simple form. Since our goal is obtaining a generalization of the Sjamaar-Lerman stratification theorem without assuming the existence of a standard momentum map and by replacing it by the cylinder valued momentum map we need to obtain a normal form for this object.

The original normal form construction in the papers by Marle and by Guillemin and Sternberg uses very strongly the existence of a standard momentum map, a hypothesis that in our setup we are not allowed to assume. We are consequently obliged to use a generalization of their construction obtained by the authors in [12] where it is shown that the symplectic version of the Slice Theorem is always available regardless of the existence of a standard momentum map. A similar generalization using different techniques has been independently obtained in [16].

In the following paragraphs we quickly review this symplectic version of the Slice Theorem since the normal form for the cylinder valued momentum map will be formulated in the coordinates provided by it. We refer to [12] or [13] for a detailed self-contained presentation of this result.

In this section we will work on a connected and paracompact symplectic manifold (M, ω) acted properly and symplectically upon by the Lie group G with Lie algebra $\mathfrak{g}$. The first step in the construction of the symplectic slice theorem is the splitting of the Lie algebra $\mathfrak{g}$ of G into three parts. The first summand is defined by

$$\mathfrak{k} := \{\xi \in \mathfrak{g} \mid \xi_M(m) \in (\mathfrak{g} \cdot m)^{\omega(m)}\}, \tag{3.1}$$

where $m \in M$ is the point around whose G-orbit we want to construct the symplectic slice. The set $\mathfrak{k}$ is clearly a vector subspace of $\mathfrak{g}$ that contains the Lie algebra $\mathfrak{g}_m$ of the isotropy subgroup G_m of the point $m \in M$. Hence we can fix an Ad_{G_m}–invariant inner product $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ on $\mathfrak{g}$ (given that the action is by hypothesis proper, the isotropy subgroup G_m is compact and hence such an inner product is always available) and write

$$\mathfrak{k} = \mathfrak{g}_m \oplus \mathfrak{m} \quad \text{and} \quad \mathfrak{g} = \mathfrak{g}_m \oplus \mathfrak{m} \oplus \mathfrak{q}, \tag{3.2}$$

where $\mathfrak{m}$ is the $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$–orthogonal complement of $\mathfrak{g}_m$ in $\mathfrak{k}$ and $\mathfrak{q}$ is the $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$–orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$. The splittings in (3.2) induce similar ones on the duals

$$\mathfrak{k}^* = \mathfrak{g}_m^* \oplus \mathfrak{m}^* \quad \text{and} \quad \mathfrak{g}^* = \mathfrak{g}_m^* \oplus \mathfrak{m}^* \oplus \mathfrak{q}^*. \tag{3.3}$$

Each of the spaces in this decomposition should be understood as the set of covectors in $\mathfrak{g}^*$ that can be written as $\langle \xi, \cdot \rangle_{\mathfrak{g}}$, with ξ in the corresponding subspace. For example, $\mathfrak{q}^* = \{\langle \xi, \cdot \rangle_{\mathfrak{g}} \mid \xi \in \mathfrak{q}\}$.

Let now $\ll \cdot, \cdot \gg$ be a G_m–invariant inner product in T_mM (available again by the compactness of G_m). Define V as the orthogonal complement to $\mathfrak{g} \cdot m \cap (\mathfrak{g} \cdot m)^{\omega(m)} = \mathfrak{k} \cdot m$ in $(\mathfrak{g} \cdot m)^{\omega(m)}$ with respect to $\ll \cdot, \cdot \gg$, that is:

$$(\mathfrak{g} \cdot m)^{\omega(m)} = \mathfrak{g} \cdot m \cap (\mathfrak{g} \cdot m)^{\omega(m)} \oplus V = \mathfrak{k} \cdot m \oplus V.$$

The subspace V is a symplectic G_m–invariant subspace of $(T_mM, \omega(m))$ such that $V \cap \mathfrak{q} \cdot m = \{0\}$. Any such space V is called a ***symplectic normal space*** at m. Since the G_m–action on $(V, \omega(m)|_V)$ is linear and symplectic it has an associated standard equivariant momentum map $\mathbf{J}_V : V \to \mathfrak{g}_m^*$ given by $\langle \mathbf{J}_V(v), \eta \rangle = \frac{1}{2}\omega(m)(\xi_V(v), v)$. The proof of the following two results can be found in [12, 13].

Proposition 3.1 (The symplectic tube) *Let (M, ω) be a connected paracompact symplectic manifold and G a Lie group acting properly and canonically on it. Let $m \in M$, V be a symplectic normal space at m, and $\mathfrak{m} \subset \mathfrak{g}$ the subspace introduced in the splitting* (3.2). *Then there exist G_m–invariant neighborhoods $\mathfrak{m}_r^*$ and V_r of the origin in $\mathfrak{m}^*$ and V, respectively, such that the twisted product*

$$Y_r := G \times_{G_m} (\mathfrak{m}_r^* \times V_r) \tag{3.4}$$

is a symplectic manifold with the two–form ω_{Y_r} defined by:

$$\begin{aligned} &\omega_{Y_r}([g, \rho, v])(T_{(g,\rho,v)}\pi(T_eL_g(\xi_1), \alpha_1, u_1), T_{(g,\rho,v)}\pi(T_eL_g(\xi_2), \alpha_2, u_2)) \\ &\quad := \langle \alpha_2 + T_v\mathbf{J}_V(u_2), \xi_1 \rangle - \langle \alpha_1 + T_v\mathbf{J}_V(u_1), \xi_2 \rangle + \langle \rho + \mathbf{J}_V(v), [\xi_1, \xi_2] \rangle \\ &\qquad + \Psi(m)(\xi_1, \xi_2) + \omega(m)(u_1, u_2), \end{aligned} \tag{3.5}$$

where $\Psi : M \to Z^2(\mathfrak{g})$ is the Chu map associated to the G–action on (M, ω), $\pi : G \times (\mathfrak{m}_r^* \times V_r) \to G \times_{G_m} (\mathfrak{m}_r^* \times V_r)$ is the projection, $[g, \rho, v] \in Y_r$, $\xi_1, \xi_2 \in \mathfrak{g}$, $\alpha_1, \alpha_2 \in \mathfrak{m}^*$, and $u_1, u_2 \in V$.

The Lie group G acts canonically on (Y_r, ω_{Y_r}) by $g \cdot [h, \eta, v] := [gh, \eta, v]$, for any $g \in G$ and any $[h, \eta, v] \in Y_r$.

*In the sequel will refer to the symplectic manifold (Y_r, ω_{Y_r}) as a **symplectic tube** of (M, ω) at the point m.*

Theorem 3.1 (Symplectic Slice Theorem) *Let (M, ω) be a symplectic manifold and let G be a Lie group acting properly and canonically on M. Let $m \in M$ and let (Y_r, ω_{Y_r}) be the G–symplectic tube at that point constructed in Proposition* 3.1. *Then there is a G–invariant neighborhood U of m in M and a G–equivariant symplectomorphism $\phi : U \to Y_r$ satisfying $\phi(m) = [e, 0, 0]$.*

We now provide an expression in the symplectic tube for the cylinder valued momentum map. This is what we call the normal form for the cylinder valued momentum map. The proof of the following theorem can be found in Appendix 6.2.

Theorem 3.2 (Normal form for the cylinder valued momentum map) *Let (M, ω) be a connected paracompact symplectic manifold acted properly and canonically upon by the connected Lie group G. Let $m \in M$ and (Y_r, ω_{Y_r}) be a symplectic tube at m that models a G–invariant neighborhood U of the orbit*

$G\cdot m$ via the G–equivariant symplectomorphism $\phi:(Y_r,\omega_{Y_r})\rightarrow(U,\omega|_U)$. Let $\mathbf{K}:M\rightarrow\mathfrak{g}^*/\overline{\mathcal{H}}$ be a cylinder valued momentum map associated to the G–action on M with non-equivariance one-cocycle $\sigma:G\rightarrow\mathfrak{g}^*/\overline{\mathcal{H}}$. Then for any $[g,\rho,v]\in Y_r$ we have

$$\begin{aligned}\mathbf{K}(\phi[g,\rho,v]) &= \Theta_g\left(\mathbf{K}(m)+\pi_C(\rho+\mathbf{J}_V(v)\right) &(3.6)\\ &= \Theta_g\left(\mathbf{K}(m)\right)+\pi_C\left(\mathrm{Ad}^*_{g^{-1}}(\rho+\mathbf{J}_V(v))\right) &(3.7)\end{aligned}$$

where $\pi_C:\mathfrak{g}^*\rightarrow\mathfrak{g}^*/\overline{\mathcal{H}}$ is the projection and $\Theta:G\times\mathfrak{g}^*/\overline{\mathcal{H}}\rightarrow\mathfrak{g}^*/\overline{\mathcal{H}}$ is the affine action associated to the non-equivariance one-cocycle σ.

Remark 3.1 A straightforward verification shows that the expression (3.6) reduces to the Marle–Guillemin–Sternberg normal form [8, 9, 5] in the presence of a standard momentum map. Indeed, in that particular case, π_C is just the identity and $\Theta:G\times\mathfrak{g}^*\rightarrow\mathfrak{g}^*$ is the affine action associated to the non-equivariance one-cocycle σ of the standard momentum map in question.

4 The stratification theorem

In this section we state in a detailed fashion and prove the main result of the paper announced in the introduction. We start by carefully describing the properties of the spaces that will constitute the pieces of the symplectic stratification of the reduced space.

Proposition 4.1 (Singular symplectic point strata) *Let (M,ω) be a connected paracompact symplectic manifold and G a Lie group acting properly and symplectically on it with closed Hamiltonian holonomy $\mathcal{H}$. Let $\mathbf{K}:M\rightarrow\mathfrak{g}^*/\mathcal{H}$ be a cylinder valued momentum map for this action with associated non-equivariance one-cocycle $\sigma:M\rightarrow\mathfrak{g}^*/\mathcal{H}$. Let $[\mu]\in\mathfrak{g}^*/\mathcal{H}$, $G_{[\mu]}$ the isotropy subgroup of $[\mu]\in\mathfrak{g}^*/\mathcal{H}$ with respect to the affine action $\Theta:G\times\mathfrak{g}^*/\mathcal{H}\rightarrow\mathfrak{g}^*/\mathcal{H}$ determined by σ, and let $H\subset G$ be an isotropy subgroup of the G–action on M. Let M_H^z be the connected component of the H–isotropy type manifold M_H that contains a given element $z\in M$ such that $\mathbf{K}(z)=[\mu]$ and let $G_{[\mu]}M_H^z$ be its $G_{[\mu]}$–saturation. Then the following hold:*

(i) *The set $\mathbf{K}^{-1}([\mu])\cap G_{[\mu]}M_H^z$ is a submanifold of M.*

(ii) *The set $M_{[\mu]}^{(H)}:=[\mathbf{K}^{-1}([\mu])\cap G_{[\mu]}M_H^z]/G_{[\mu]}$ has a unique quotient differentiable structure such that the canonical projection*

$$\pi_{[\mu]}^{(H)}:\mathbf{K}^{-1}([\mu])\cap G_{[\mu]}M_H^z\longrightarrow M_{[\mu]}^{(H)}$$

is a surjective submersion.

(iii) *There is a unique symplectic structure $\omega_{[\mu]}^{(H)}$ on $M_{[\mu]}^{(H)}$ characterized by*

$$i_{[\mu]}^{(H)\,*}\omega=\pi_{[\mu]}^{(H)\,*}\omega_{[\mu]}^{(H)},$$

where $i_{[\mu]}^{(H)}:\mathbf{K}^{-1}(\mu)\cap G_{[\mu]}M_H^z\hookrightarrow M$ is the natural inclusion. The pairs $(M_{[\mu]}^{(H)},\omega_{[\mu]}^{(H)})$ will be called ***singular symplectic point strata****.*

(iv) *Let $h\in C^\infty(M)^G$ be a G–invariant Hamiltonian. Then the flow F_t of X_h leaves the connected components of $\mathbf{K}^{-1}([\mu])\cap G_{[\mu]}M_H^z$ invariant and commutes with the $G_{[\mu]}$–action, so it induces a flow F_t^μ on $M_{[\mu]}^{(H)}$ that is characterized by*

$$\pi_{[\mu]}^{(H)}\circ F_t\circ i_{[\mu]}^{(H)}=F_t^\mu\circ\pi_{[\mu]}^{(H)}.$$

(v) *The flow F_t^μ is Hamiltonian on $M_{[\mu]}^{(H)}$, with* ***reduced Hamiltonian*** *function $h_{[\mu]}^{(H)}:M_{[\mu]}^{(H)}\rightarrow\mathbb{R}$ defined by*

$$h_{[\mu]}^{(H)}\circ\pi_{[\mu]}^{(H)}=h\circ i_{[\mu]}^{(H)}.$$

The vector fields X_h and $X_{h_{[\mu]}^{(H)}}$ are $\pi_{[\mu]}^{(H)}$–related.

(vi) *Let $k : M \to \mathbb{R}$ be another G–invariant function. Then $\{h, k\}$ is also G–invariant and*

$$\{h,k\}^{(H)}_{[\mu]} = \{h^{(H)}_{[\mu]}, k^{(H)}_{[\mu]}\}_{M^{(H)}_{[\mu]}}$$

where $\{\ ,\ \}_{M^{(H)}_{[\mu]}}$ denotes the Poisson bracket induced by the symplectic structure on $M^{(H)}_{[\mu]}$.

Theorem 4.1 (Stratification theorem) *Let (M, ω) be a connected paracompact symplectic manifold and G a Lie group acting properly and symplectically on it with closed Hamiltonian holonomy $\mathcal{H}$. Let $\mathbf{K} : M \to \mathfrak{g}^*/\mathcal{H}$ be a cylinder valued momentum map for this action with associated non-equivariance one-cocycle $\sigma : M \to \mathfrak{g}^*/\mathcal{H}$. Let $[\mu] \in \mathfrak{g}^*/\mathcal{H}$. The manifolds $M^{(H)}_{[\mu]} := [\mathbf{K}^{-1}([\mu]) \cap G_{[\mu]} M^z_H]/G_{[\mu]}$, with H any isotropy subgroup of the G-action on M and $z \in M_H$, form a symplectic Whitney stratification of the quotient $M_{[\mu]} := \mathbf{K}([\mu])/G_{[\mu]}$. The quotient $M_{[\mu]}$ is a cone space when considered as a stratified space with strata $M^{(H)}_{[\mu]}$.*

The strategy to prove the two results that we just stated consists of using the normal form for the cylinder valued momentum map presented in Theorem 3.2 to obtain a local characterization of the reduced space $M_{[\mu]} := \mathbf{K}^{-1}([\mu])/G_{[\mu]}$ and of the strata $M^{(H)}_{[\mu]}$; this reduces the proof of these results to those that have already been carried out for the situation in which there is a standard momentum map available. The following result is crucial to this method.

Proposition 4.2 *Let (M, ω) be a connected paracompact symplectic manifold and G a Lie group acting properly and symplectically on it with closed Hamiltonian holonomy $\mathcal{H}$. Let $\mathbf{K} : M \to \mathfrak{g}^*/\mathcal{H}$ be a cylinder valued momentum map for this action with associated non-equivariance one-cocycle $\sigma : M \to \mathfrak{g}^*/\mathcal{H}$. Let $m \in M$, $[\mu] = \mathbf{K}(m)$, and $(Y_r = G \times_{G_m} (\mathfrak{m}^*_r \times V_r), \omega_{Y_r})$ be a symplectic tube around that point. Then there is a $G_{[\mu]}$–invariant open neighborhood $Y_0 \subset Y_r$ of the orbit $G_{[\mu]} \cdot [e, 0, 0]$ such that*

$$\mathbf{K}^{-1}_{Y_r}([\mu]) \cap Y_0 = \left\{[g, 0, v] \in Y_0 \mid g \in G_{[\mu]},\ \textit{and}\ \mathbf{K}_V(v) = 0\right\}.$$

The symbol $G_{[\mu]}$ denotes the isotropy subgroup of $[\mu] \in \mathfrak{g}^*/\mathcal{H}$ with respect to the affine action $\Theta : G \times \mathfrak{g}^*/\mathcal{H} \to \mathfrak{g}^*/\mathcal{H}$ determined by σ.

Proof. Since by Theorem 3.2 the cylinder valued momentum map takes in the tube the form $\mathbf{K}([g, \rho, v]) = \Theta_g(\mathbf{K}(m)) + \pi_C\left(\mathrm{Ad}^*_{g^{-1}}(\rho + \mathbf{J}_V(v))\right)$, $[g, \rho, v] \in G \times_{G_m} (\mathfrak{m}^*_r \times V_r)$, we can write $\mathbf{K} = \epsilon \circ b$, where

$$\begin{array}{rccc} b: & G \times_{G_m} (\mathfrak{m}^*_r \times V_r) & \longrightarrow & G \times_{G_m} \mathfrak{k}^* \\ & [g, \rho, v] & \longmapsto & [g, \rho + \mathbf{J}_V(v)], \end{array}$$

$$\begin{array}{rccc} \epsilon: & G \times_{G_m} \mathfrak{k}^* & \longrightarrow & \mathfrak{g}^*/\mathcal{H} \\ & [g, \nu] & \longmapsto & \Theta_g(\mathbf{K}(m) + \nu) = \pi_C(\mathrm{Ad}^*_{g^{-1}}(\mathbf{K}(m) + \nu)) + \sigma(g). \end{array}$$

Notice now that since $\mathcal{H}$ is closed in $\mathfrak{g}^*$ then $\ker T_m\mathbf{K} = (\mathfrak{g} \cdot m)^{\omega(m)}$, for any $m \in M$ and hence $\mathfrak{k} = \mathfrak{g}_{[\mu]}$, which implies that we can make the identification $\mathfrak{k}^* = \mathfrak{g}^*_{[\mu]}$. Additionally, the projection π_C is a local diffeomorphism and hence $T_\mu\pi_C$ is an isomorphism for any $\mu \in \mathfrak{g}^*$.

Having these points in mind it can be shown that the map $T_{[e,0]}\epsilon : T_{[e,0]}(G \times_{G_m} \mathfrak{g}^*_{[\mu]}) \longrightarrow T_{[\mu]}(\mathfrak{g}^*/\mathcal{H})$ is surjective. Indeed, a straightforward computation shows that

$$T_{[e,0]}\epsilon(T_{[e,0]}(G \times_{G_m} \mathfrak{g}^*_{[\mu]}) = \mathfrak{g} \cdot (\mathbf{K}(m)) + T_0\pi_C \cdot \mathfrak{g}^*_{[\mu]} = \mathfrak{g} \cdot [\mu] + T_0\pi_C \cdot \mathfrak{g}^*_{[\mu]}.$$

Since $\dim \mathfrak{g} \cdot [\mu] + \dim \mathfrak{g}^*_{[\mu]} = \dim \mathfrak{g} = \dim(\mathfrak{g}^*/\mathcal{H})$, it suffices to show that $\mathfrak{g} \cdot [\mu] \cap T_0\pi_C \cdot \mathfrak{g}^*_{[\mu]} = \{0\}$. Hence, consider $v \in \mathfrak{g} \cdot [\mu] \cap T_0\pi_C \cdot \mathfrak{g}^*_{[\mu]}$. There exist elements $\xi \in \mathfrak{g}$ and $\rho \in \mathfrak{g}^*_{[\mu]}$ such that

$$v = \xi_{\mathfrak{g}^*/\mathcal{H}}([\mu]) = T_0\pi_C \cdot \rho. \tag{4.1}$$

We are going to show that $v = 0$. Let $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ be the Ad_{G_m}-invariant inner product in $\mathfrak{g}$ that we used in Section 3 to write down the splittings $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{q} = \mathfrak{g}_{[\mu]} \oplus \mathfrak{q}$ and $\mathfrak{g}^* = \mathfrak{g}^*_{[\mu]} \oplus \mathfrak{q}^*$. Let $\lambda \in \mathfrak{g}_{[\mu]}$ be the element such that $\rho = \langle \lambda, \cdot \rangle_{\mathfrak{g}}$. Then, by (2.3) we have

$$\begin{aligned} \|\lambda\|^2_{\mathfrak{g}} := \langle \lambda, \lambda \rangle_{\mathfrak{g}} &= \langle \rho, \lambda \rangle = \langle (T_0\pi_C)^{-1} \xi_{\mathfrak{g}^*/\mathcal{H}}([\mu]), \lambda \rangle = -\langle \Psi(m)(\xi, \cdot), \lambda \rangle = -\omega(m)(\xi_M(m), \lambda_M(m)) \\ &= \omega(m)(\lambda_M(m), \xi_M(m)) = \langle \Psi(m)(\lambda, \cdot), \xi \rangle = \langle (T_0\pi_C)^{-1} \lambda_{\mathfrak{g}^*/\mathcal{H}}([\mu]), \xi \rangle = 0, \end{aligned}$$

where the last equality follows from the fact that $\lambda \in \mathfrak{g}_{[\mu]}$. This shows that $v = 0$ and hence that $T_{[e,0]}\epsilon$ is surjective.

Let $\phi : G \times \left(G \times_{G_m} \mathfrak{g}^*_{[\mu]} \right) \to \left(G \times_{G_m} \mathfrak{g}^*_{[\mu]} \right)$ be the left action defined by $\phi(g, [h, \nu]) := [gh, \nu]$, $g \in G$, $[h, \nu] \in G \times_{G_m} \mathfrak{g}^*_{[\mu]}$. By its very definition, the map ϵ is equivariant, that is,

$$\Theta_g \circ \epsilon = \epsilon \circ \phi_g, \text{ for any } g \in G. \tag{4.2}$$

Therefore, if $[g, 0] \in G \times_{G_m} \mathfrak{g}^*_{[\mu]}$ we have $T_{[g,0]}\epsilon = T_\mu \Theta_g \circ T_{[e,0]}\epsilon \circ T_{[g,0]}\phi_{g^{-1}}$ and hence, as $T_\mu \Theta_g$ and $T_{[g,0]}\phi_{g^{-1}}$ are isomorphisms and $T_{[e,0]}\epsilon$ is surjective, it follows that $T_{[g,0]}\epsilon$ is also surjective. Thus ϵ is a submersion at all points of the form $[g, 0]$, for any $g \in G$. Moreover, as submersivity is an open condition, there is an open neighborhood $U_{[e,0]}$ of the point $[e, 0]$ such that $T_{[g,\nu]}\epsilon$ is onto, for any $[g, \nu] \in U_{[e,0]}$. The equivariance property (4.2) of ϵ implies that the open G–invariant neighborhood $U := G \cdot U_{[e,0]} = \bigcup_{g \in G} \phi_g(U_{[e,0]})$ of the orbit $G \cdot [e, 0]$ has also the same property.

By the Submersion Theorem, $\epsilon^{-1}([\mu]) \cap U$ is a submanifold of $G \times_{G_m} \mathfrak{g}^*_{[\mu]}$ of dimension $\dim G_{[\mu]} - \dim G_m$. However, the submanifold $G_{[\mu]} \times_{G_m} \{0\}$ of $G \times_{G_m} \mathfrak{g}^*_{[\mu]}$ is included in $\epsilon^{-1}([\mu]) \cap U$ and has also dimension $\dim G_{[\mu]} - \dim G_m$. Therefore $G_{[\mu]} \times_{G_m} \{0\}$ is an open submanifold of $\epsilon^{-1}([\mu]) \cap U$.

We are now going to prove that there exists an open $G_{[\mu]}$–invariant subset V of U such that

$$G_{[\mu]} \times_{G_m} \{0\} = \epsilon^{-1}([\mu]) \cap V. \tag{4.3}$$

We start by noticing that because $\epsilon^{-1}([\mu]) \cap U$ is an embedded submanifold of $G \times_{G_m} \mathfrak{g}^*_{[\mu]}$, the sets $\epsilon^{-1}([\mu]) \cap U \cap W$, with W open in $G \times_{G_m} \mathfrak{g}^*_{[\mu]}$, form a basis of its topology. The same can be said about the family $\mathcal{B} := \{\epsilon^{-1}([\mu]) \cap U_i \mid i \in I, U_i \subset U, U_i \text{ open in } G \times_{G_m} \mathfrak{g}^*_{[\mu]}\}$. Consequently, as $G_{[\mu]} \times_{G_m} \{0\}$ is an open subset of $\epsilon^{-1}([\mu]) \cap U$, there exists a subset $J \subset I$ such that

$$G_{[\mu]} \times_{G_m} \{0\} = \bigcup_{j \in J} \epsilon^{-1}([\mu]) \cap U_j. \tag{4.4}$$

Since U is G–invariant, the saturations $G_{[\mu]} \cdot U_j$ are open subsets of U. Let us show that

$$G_{[\mu]} \times_{G_m} \{0\} = \bigcup_{j \in J} \epsilon^{-1}([\mu]) \cap G_{[\mu]} \cdot U_j. \tag{4.5}$$

Indeed, the inclusion $G_{[\mu]} \times_{G_m} \{0\} \subset \bigcup_{j \in J} \epsilon^{-1}([\mu]) \cap G_{[\mu]} \cdot U_j$ is obvious because of (4.4). Conversely, given a point $h \cdot z \in \epsilon^{-1}([\mu]) \cap G_{[\mu]} \cdot U_j$ with $h \in G_{[\mu]}$ and $z \in U_j$ and such that $\epsilon(h \cdot z) = [\mu]$, the equivariance property (4.2) of ϵ implies that $\Theta_h(\epsilon(z)) = [\mu]$ and hence $\epsilon(z) = \Theta_{h^{-1}}([\mu]) = [\mu]$ because $h \in G_{[\mu]}$. Thus, $z \in \epsilon^{-1}([\mu]) \cap U_j$ and by (4.4) it can be written as $z = [g, 0]$ with $g \in G_{[\mu]}$. Finally, $h \cdot z = [hg, 0] \in G_{[\mu]} \times_{G_m} \{0\}$, as required.

Expression (4.5) implies that $G_{[\mu]} \times_{G_m} \{0\} = \bigcup_{j \in J} \epsilon^{-1}([\mu]) \cap G_{[\mu]} \cdot U_j = \epsilon^{-1}([\mu]) \bigcap \left(\bigcup_{j \in J} G_{[\mu]} \cdot U_j \right)$ and therefore (4.3) holds by taking $V = \bigcup_{j \in J} G_{[\mu]} \cdot U_j$ which is obviously $G_{[\mu]}$–invariant and contains $G_{[\mu]} \cdot [e, 0]$.

Define $Y_0 := b^{-1}(V) \supset b^{-1}(G_{[\mu]} \cdot [e, 0]) \supset G_{[\mu]} \cdot [e, 0, 0]$. The set Y_0 is open in Y_r by smoothness of b and is $G_{[\mu]}$–invariant by G–equivariance of b (the G–action on the source and target is on the first factor only). Moreover,

$$\begin{aligned} \mathbf{K}^{-1}([\mu]) \cap Y_0 &= b^{-1}(\epsilon^{-1}([\mu])) \cap b^{-1}(V) = b^{-1}(G_{[\mu]} \times_{G_m} \{0\}) \\ &= \{[g, \rho, v] \in Y_0 \mid g \in G_{[\mu]}, \ \rho = 0 \text{ and } \mathbf{J}_V(v) = 0\} \end{aligned}$$

and the claim of the Proposition holds. ■

With this result at hand the proofs of Proposition 4.1 and Theorem 4.1 mimic the analogous results available for the case in which a standard momentum map exists. These are the proofs of Theorems 8.1.1 and 8.3.2 in [13]. Once Proposition 4.2 is available, the proofs of Proposition 4.1 and Theorem 4.1 are virtual copies of the the proofs of the afforementioned theorems by replacing everywhere the isotropy groups G_μ by $G_{[\mu]}$.

5 Reduction in the presence of a group valued momentum map

The cylinder valued momentum maps are closely related to the ***Lie group valued momentum maps*** introduced by [11, 4, 7, 6], and [1]. We give the definition of these objects only for Abelian symmetry groups because in the non-Abelian case these momentum maps are defined on spaces that are not symplectic (they are referred to as ***quasi Hamiltonian spaces***) thereby leaving the category on which we focus in this paper.

Definition 5.1 *Let G be an Abelian Lie group whose Lie algebra $\mathfrak{g}$ acts canonically on the symplectic manifold (M, ω). Let $(\cdot, \cdot)$ be some bilinear symmetric non degenerate form on the Lie algebra $\mathfrak{g}$. The map $\mathbf{J} : M \to G$ is called a G–**valued momentum map** for the $\mathfrak{g}$–action on M whenever*

$$\mathbf{i}_{\xi_M}\omega(m) \cdot v_m = \left(T_m(L_{\mathbf{J}(m)^{-1}} \circ \mathbf{J})(v_m), \xi\right), \tag{5.1}$$

for any $\xi \in \mathfrak{g}$, $m \in M$, and $v_m \in T_mM$.

In this section we shall see that existence of a Lie group valued momentum map automatically guarantees the closedness in $\mathfrak{g}^*$ of the Hamiltonian holonomy $\mathcal{H}$ hence making valid the stratification theorem 4.1. Moreover, the close relation between these two momentum maps that we will discuss in the following paragraphs will provide in certain circumstances a singular reduction theorem for the Lie group valued momentum map. We start with a very suggestive proposition that states that any cylinder valued momentum map associated to an Abelian Lie algebra action whose corresponding holonomy group is closed can be understood as Lie group valued momentum map. The proof is provided in the appendix.

Proposition 5.1 *Let (M, ω) be a connected paracompact symplectic manifold and $\mathfrak{g}$ an Abelian Lie algebra acting canonically on it. Let $\mathcal{H} \subset \mathfrak{g}^*$ be the Hamiltonian holonomy of the action and $(\cdot, \cdot) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ some bilinear symmetric non degenerate form on $\mathfrak{g}$. Let $f : \mathfrak{g} \to \mathfrak{g}^*$ be the isomorphism given by $\xi \longmapsto (\xi, \cdot)$, $\xi \in \mathfrak{g}$ and $\mathcal{T} := f^{-1}(\mathcal{H})$. The map f induces an Abelian group isomorphism $\bar{f} : \mathfrak{g}/\mathcal{T} \to \mathfrak{g}^*/\mathcal{H}$ by $\bar{f}(\xi + \mathcal{T}) := (\xi, \cdot) + \mathcal{H}$. Suppose that $\mathcal{H}$ is closed in $\mathfrak{g}^*$ and define $\mathbf{J} := \bar{f}^{-1} \circ \mathbf{K} : M \to \mathfrak{g}/\mathcal{T}$, where $\mathbf{K} : M \to \mathfrak{g}^*/\mathcal{H}$ is a cylinder valued momentum map for the $\mathfrak{g}$–action on (M, ω). Then*

$$\mathbf{i}_{\xi_M}\omega(m) \cdot v_m = \left(T_m(L_{\mathbf{J}(m)^{-1}} \circ \mathbf{J})(v_m), \xi\right), \tag{5.2}$$

for any $\xi \in \mathfrak{g}$ and $v_m \in T_mM$. Consequently, the map $\mathbf{J} : M \to \mathfrak{g}/\mathcal{T}$ constitutes a $\mathfrak{g}/\mathcal{T}$–valued momentum map for the canonical action of the Lie algebra $\mathfrak{g}$ of $(\mathfrak{g}/\mathcal{T}, +)$ on (M, ω).

The previous proposition shows how cylinder valued momentum maps can be viewed as a Lie group valued momentum maps. Now we shall focus on the converse relation, that is, we shall study the implications of the existence of a Lie group valued momentum map for the Hamiltonian holonomy and we will isolate hypotheses that allow us to express the Lie group valued momentum map in terms of a cylinder valued momentum map. The proof is also given in the appendix.

Theorem 5.1 *Let (M, ω) be a connected paracompact symplectic manifold and $\mathfrak{g}$ an Abelian Lie algebra acting canonically on it. Let $\mathcal{H} \subset \mathfrak{g}^*$ be the Hamiltonian holonomy of the $\mathfrak{g}$–action and let $(\cdot, \cdot) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ be a bilinear symmetric non-degenerate form on $\mathfrak{g}$. Let $f : \mathfrak{g} \to \mathfrak{g}^*$, $\bar{f} : \mathfrak{g}/\mathcal{T} \to \mathfrak{g}^*/\mathcal{H}$, and $\mathcal{T} := f^{-1}(\mathcal{H})$ be as in the statement of Proposition 5.1. Let G be a connected Abelian Lie group whose Lie algebra is $\mathfrak{g}$ and suppose that there exists a G–valued momentum map $\mathbf{A} : M \to G$ associated to the $\mathfrak{g}$–action whose definition uses the form $(\cdot, \cdot)$.*

(i) *If* $\exp : \mathfrak{g} \to G$ *is the exponential map, then*

$$\mathcal{H} \subset f(\ker \exp). \tag{5.3}$$

(ii) $\mathcal{H}$ *is closed in* $\mathfrak{g}^*$.

Let $\mathbf{J} := \bar{f}^{-1} \circ \mathbf{K} : M \to \mathfrak{g}/\mathcal{T}$, *where* $\mathbf{K} : M \to \mathfrak{g}^*/\mathcal{H}$ *is a cylinder valued momentum map for the* $\mathfrak{g}$*–action on* (M, ω). *If* $f(\ker \exp) \subset \mathcal{H}$ *then* $\mathbf{J} : M \to \mathfrak{g}/\mathcal{T} = \mathfrak{g}/\ker \exp \simeq G$ *is a* G*–valued momentum map that differs from* $\mathbf{A}$ *by a constant in* G.

Conversely, if $\mathcal{H} = f(\ker \exp)$ *then* $\mathbf{J} : M \to \mathfrak{g}/\ker \exp \simeq G$ *is a* G*–valued momentum map.*

Remark 5.1 The presence of a Lie group valued momentum map associated to a canonical Lie algebra action does NOT imply the reverse inclusion in (5.3). A simple example that illustrates this statement is the canonical action of a two torus $\mathbb{T}^2$ on itself via

$$(e^{i\phi_1}, e^{i\phi_2}) \cdot (e^{i\theta_1}, e^{i\theta_2}) := (e^{i(\theta_1+\phi_1)}, e^{i\theta_2}),$$

where we consider the torus as a symplectic manifold with the area form. A straightforward computation shows that the surface

$$\widetilde{\mathbb{T}^2} := \left\{ ((e^{i\theta_1}, e^{i\theta_2}), (\theta_2, 0)) \in \mathbb{T}^2 \times \mathbb{R}^2 \mid \theta_1, \theta_2 \in \mathbb{R} \right\},$$

is the holonomy bundle containing the point $((e, e), (0, 0)) \in \mathbb{T}^2 \times \mathbb{R}^2$ associated to the connection that defines the corresponding cylinder valued momentum map. This immediately shows that $\mathcal{H} = \mathbb{Z} \times \{0\}$ while $f(\ker \exp) = \mathbb{Z} \times \mathbb{Z}$ which is clearly not contained in $\mathcal{H}$.

6 Appendices

6.1 The flatness of α

The vertical subbundle $V \subset T(M \times \mathfrak{g}^*)$ of $\pi : M \times \mathfrak{g}^* \to M$ is given for any $(m, \mu) \in M \times \mathfrak{g}^*$ by

$$V(m, \mu) := \{(0, \rho) \in T_{(m,\mu)}(M \times \mathfrak{g}^*) \mid \rho \in \mathfrak{g}^*\}. \tag{6.1}$$

By definition, the horizontal subspace $H(m, \mu)$ determined by α at the point $(m, \mu) \in M \times \mathfrak{g}^*$ is given by

$$H(m, \mu) = \{(v_m, \nu) \in T_{(m,\mu)}(M \times \mathfrak{g}^*) \mid (\mathbf{i}_{\xi_M}\omega)(m) \cdot v_m - \langle \nu, \xi \rangle = 0, \qquad \text{for all} \quad \xi \in \mathfrak{g}\}. \tag{6.2}$$

Consequently, given any vector $(v_m, \nu) \in T_{(m,\mu)}(M \times \mathfrak{g}^*)$, its horizontal $(v_m, \nu)^H$ and vertical $(v_m, \nu)^V$ parts are such that

$$(v_m, \nu)^H = (v_m, \rho) \quad \text{and} \quad (v_m, \nu)^V = (0, \rho'),$$

where $\rho, \rho' \in \mathfrak{g}^*$ are uniquely determined by the relations

$$\langle \rho, \xi \rangle = (\mathbf{i}_{\xi_M}\omega)(m) \cdot v_m \quad \text{and} \quad \rho' = \nu - \rho, \quad \text{for any} \quad \xi \in \mathfrak{g}.$$

We now compute the curvature form Ω associated to α. Let $(m, \mu) \in M \times \mathfrak{g}^*$, $v_m, u_m \in T_m M$, $\xi \in \mathfrak{g}$, and $\nu, \rho \in \mathfrak{g}^*$ be arbitrary. By definition,

$$\langle \Omega(m, \mu)((v_m, \nu), (u_m, \rho)), \xi \rangle = \langle \mathbf{d}\alpha(m, \mu)((v_m, \nu)^H, (u_m, \rho)^H), \xi \rangle. \tag{6.3}$$

Let now (X_1, Y_1) and (X_2, Y_2) be vector fields on $M \times \mathfrak{g}^*$ such that $(X_1(m), Y_1(\mu)) = (v_m, \nu)$ and $(X_2(m), Y_2(\mu)) = (u_m, \rho)$. Using these vector fields, the right-hand side of (6.3) can be rewritten as

$$\langle (X_1, Y_1)[\alpha(X_2, Y_2)](m, \mu), \xi \rangle - \langle (X_2, Y_2)[\alpha(X_1, Y_1)](m, \mu), \xi \rangle - \langle \alpha([X_1, X_2], 0)(m, \mu), \xi \rangle. \tag{6.4}$$

Let (m_t^1, μ_t^1) and (m_t^2, μ_t^2) be the flows of (X_1, Y_1) and (X_2, Y_2), respectively. Choose Y_1 and Y_2 with flows $\mu_t^1(\mu) = \mu + t\nu$ and $\mu_t^2(\mu) = \mu + t\rho$. We can use these flows to compute

$$\begin{aligned}\langle (X_1, Y_1)[\alpha(X_2, Y_2)](m, \mu), \xi\rangle &= \left.\frac{d}{dt}\right|_{t=0} \left\langle \alpha(m_t^1(m), \mu_t^1(\mu)) \cdot (X_2(m_t^1(m)), Y_2(\mu_t^1(\mu))), \xi\right\rangle \\ &= \left.\frac{d}{dt}\right|_{t=0} \left((\mathbf{i}_{\xi_M}\omega)(m_t^1(m)) \cdot X_2(m_t^1(m)) - \left\langle Y_2(\mu_t^1(\mu)), \xi\right\rangle\right) \\ &= X_1\left[\mathbf{i}_{\xi_M}\omega(X_2)\right](m) - \left.\frac{d}{dt}\right|_{t=0} \left.\frac{d}{ds}\right|_{s=0} \langle \mu + t\nu + s\rho, \xi\rangle \\ &= X_1\left[\mathbf{i}_{\xi_M}\omega(X_2)\right](m).\end{aligned}$$

Analogously, we have $\langle (X_2, Y_2)[\alpha(X_1, Y_1)](m, \mu), \xi\rangle = X_2\left[\mathbf{i}_{\xi_M}\omega(X_1)\right](m)$. Consequently, the expression (6.4) equals

$$\begin{aligned}X_1\left[\mathbf{i}_{\xi_M}\omega(X_2)\right](m) - X_2\left[\mathbf{i}_{\xi_M}\omega(X_1)\right](m) - \mathbf{i}_{\xi_M}\omega(m)([X_1, X_2](m)) \\ = \mathbf{d}(\mathbf{i}_{\xi_M}\omega)(m)(X_1(m), X_2(m)) \\ = (\pounds_{\xi_M}\omega)(m)(X_1(m), X_2(m)) - (\mathbf{i}_{\xi_M}\mathbf{d}\omega)(m)(X_1(m), X_2(m)) = 0,\end{aligned}$$

which guarantees the flatness of α.

6.2 Proof of Theorem 3.2

One of the main tools that we will need in the proof of the theorem is the connection introduced in the following proposition.

Proposition 6.1 *Let G be a connected Lie group acting canonically on the connected paracompact symplectic manifold (M, ω). For $m \in M$ let $G_m \subset G$ be its isotropy subgroup. Let $G/G_m \times \mathfrak{g}^* \to G/G_m$ be the projection considered as a trivial principal $(\mathfrak{g}^*, +)$–bundle where the action of the vector group $(\mathfrak{g}^*, +)$ on $G/G_m \times \mathfrak{g}^*$ is given by $R_\tau(gG_m, \mu) := (gG_m, \mu - \tau)$, $gG_m \in G/G_m$, $\mu, \tau \in \mathfrak{g}^*$. For any $gG_m \in G/G_m$ and $\mu \in \mathfrak{g}^*$ define*

$$H_{G/G_m}(gG_m, \mu) := \{(T_{gG_m}\pi_{G_m}(T_eL_g(\xi)), -\mathrm{Ad}^*_{g^{-1}}(\Psi(m)(\xi, \cdot))) \mid \xi \in \mathfrak{g}\}, \tag{6.5}$$

where $\pi_{G_m} : G \to G/G_m$ denotes the projection and $\Psi : M \to Z^2(\mathfrak{g})$ is the Chu map associated to the $\mathfrak{g}$–action on (M, ω). The collection of spaces $H_{G/G_m}(gG_m, \mu)$, for any $gG_m \in G/G_m$ and $\mu \in \mathfrak{g}^*$, constitutes the horizontal bundle of a principal connection α_{G/G_m} on $G/G_m \times \mathfrak{g}^* \to G/G_m$.

Proof. We start by showing that H_{G/G_m} is the graph of a smooth function. Let $F : T(G/G_m) \times \mathfrak{g}^* \to \mathfrak{g}^*$ be given by

$$F(T_{gG_m}\pi_{G_m}(T_eL_g(\xi)), \mu) := -\mathrm{Ad}^*_{g^{-1}}(\Psi(m)(\xi, \cdot)). \tag{6.6}$$

We check that this is a good definition. For any $h \in G_m$ we have

$$T_{gG_m}\pi_{G_m}(T_eL_g(\xi)) = T_{ghG_m}\pi_{G_m}(T_eL_{gh}(\mathrm{Ad}_{h^{-1}}\xi)).$$

We can use the equivariance properties of the Chu map to write

$$\begin{aligned}F(T_{ghG_m}\pi_{G_m}(T_eL_{gh}(\mathrm{Ad}_{h^{-1}}\xi)), \mu) &= -\mathrm{Ad}^*_{(gh)^{-1}}(\Psi(m)(\mathrm{Ad}_{h^{-1}}\xi, \cdot)) \\ &= -\mathrm{Ad}^*_{g^{-1}}(\Psi(m)(\xi, \cdot)) = F(T_{gG_m}\pi_{G_m}(T_eL_g(\xi)), \mu),\end{aligned}$$

which proves that F is well defined. The map F is smooth because for any $gG_m \in G/G_m$ there is a local smooth section σ of π_{G_m} defined on a local neighborhood $U \subset G/G_m$ around gG_m such that for any $v_z \in T_z(G/G_m)$, $z \in U$ we can write

$$F(v_z, \mu) = -\mathrm{Ad}^*_{\sigma(z)^{-1}}(\Psi(m)(T_{\sigma(z)}L_{\sigma(z)^{-1}}(T_z\sigma(v_z)), \cdot)),$$

which being smooth guarantees the smoothness of F. Since H_{G/G_m} is clearly the graph of F, this immediately implies that H_{G/G_m} is a smooth submanifold of $T(G/G_m \times \mathfrak{g}^*)$. Additionally, since for any $(gG_m, \mu) \in G/G_m \times \mathfrak{g}^*$ the value $H_{G/G_m}(gG_m, \mu)$ is a vector subspace of $T_{(gG_m,\mu)}(G/G_m \times \mathfrak{g}^*)$ we have proved that H_{G/G_m} is a subbundle of $T(G/G_m \times \mathfrak{g}^*)$.

We now show that H_{G/G_m} is the horizontal bundle of a principal connection α_{G/G_m} on $G/G_m \times \mathfrak{g}^* \to G/G_m$. Since the vertical bundle V_{G/G_m} associated to the projection $G/G_m \times \mathfrak{g}^* \to G/G_m$ is given by $V_{G/G_m}(gG_m, \mu) = \{(0, \nu) \mid \nu \in \mathfrak{g}^*\}$ it suffices to check the following two points:

(i) $T_{(gG_m,\mu)}(G/G_m \times \mathfrak{g}^*) = H_{G/G_m}(gG_m, \mu) \oplus V_{G/G_m}(gG_m, \mu)$, for any $(gG_m, \mu) \in G/G_m \times \mathfrak{g}^*$. First of all it is obvious that $T_{(gG_m,\mu)}(G/G_m \times \mathfrak{g}^*) = H_{G/G_m}(gG_m, \mu) + V_{G/G_m}(gG_m, \mu)$. Now let $v_{(gG_m,\mu)} = (T_{gG_m}\pi_{G_m}(T_eL_g(\xi)), -\mathrm{Ad}^*_{g^{-1}}(\Psi(m)(\xi, \cdot))) \in H_{G/G_m}(gG_m, \mu) \cap V_{G/G_m}(gG_m, \mu)$. Since $T_{gG_m}\pi_{G_m}(T_eL_g(\xi)) = 0$, it follows that $\xi \in \mathfrak{g}_m$, which implies that $\Psi(m)(\xi, \cdot) = 0$ and hence $-\mathrm{Ad}^*_{g^{-1}}(\Psi(m)(\xi, \cdot)) = 0$. Consequently, $v_{(gG_m,\mu)} = 0$, as required.

(ii) $T_{(gG_m,\mu)}R_\tau[H_{G/G_m}(gG_m, \mu)] = H_{G/G_m}(R_\tau(gG_m, \mu))$, for any $\tau \in \mathfrak{g}^*$. This equality is a trivial consequence of the identity $T_{(gG_m,\mu)}R_\tau(T_{gG_m}\pi_{G_m}(T_eL_g(\xi)), (\mu, \nu)) = (T_{gG_m}\pi_{G_m}(T_eL_g(\xi)), (\mu - \tau, \nu))$, for any $g \in G$, $\xi \in \mathfrak{g}$, and $\mu, \nu, \tau \in \mathfrak{g}^*$. $\blacktriangledown$

We continue the proof with a few other preparatory results. In the rest of this section we will denote by α and α_{Y_r} the connections that define cylinder valued momentum maps associated to the G–actions on M and Y_r, respectively. Additionally, $\mathcal{H}$ and $\mathcal{H}_{Y_r}$ will denote the corresponding Hamiltonian holonomy groups.

Lemma 6.1 *In the hypotheses of Theorem 3.2, we have that* $\mathcal{H}_{Y_r} \subset \mathcal{H}$.

Proof. Let $\mu \in \mathcal{H}_{Y_r}$. By definition, there exists a piecewise smooth loop $y : [0, 1] \to Y_r$ such that $y(0) = y(1) = [e, 0, 0]$ and whose horizontal lift $\widetilde{y}(t) = (y(t), \mu(t))$ is such that $\mu(1) - \mu(0) = \mu$. Consider the loop $\gamma := \phi \circ y : [0, 1] \to U$ such that $\gamma(0) = \gamma(1) = m$. We now check that $\widetilde{\gamma}(t) := (\gamma(t), \mu(t))$ is the horizontal lift of γ by verifying that

$$\alpha(\gamma(t), \mu(t))(\dot{\gamma}(t), \dot{\mu}(t)) = 0. \tag{6.7}$$

By the definition of the connection α, this relation is equivalent to

$$(\mathbf{i}_{\xi_M}\omega)(\gamma(t)) \cdot \dot{\gamma}(t) = \langle \dot{\mu}(t), \xi\rangle, \quad \text{for all} \quad \xi \in \mathfrak{g},$$

which, by the definition of the loop γ, can be rewritten as

$$\omega(\phi(y(t)))(\xi_M(\phi(y(t))), T_{y(t)}\phi \cdot \dot{y}(t)) = \langle \dot{\mu}(t), \xi\rangle.$$

The G–equivariance of ϕ implies that this expression amounts to

$$\omega(\phi(y(t)))(T_{y(t)}\phi \cdot \xi_{Y_r}(y(t)), T_{y(t)}\phi \cdot \dot{y}(t)) = \langle \dot{\mu}(t), \xi\rangle.$$

Since ϕ is a symplectomorphism, this equality is equivalent to

$$\omega_{Y_r}(y(t))(\xi_{Y_r}(y(t)), \dot{y}(t)) = \langle \dot{\mu}(t), \xi\rangle,$$

which is true since the curve $\widetilde{y}(t) = (y(t), \mu(t))$ is horizontal with respect to α_{Y_r}. This chain of equivalences proves that (6.7) is verified and therefore that $\mu = \mu(1) - \mu(0) \in \mathcal{H}$, as required. $\blacktriangledown$

We now consider the symplectic manifold (Y_r, ω_{Y_r}) acted canonically upon by the Lie group G and we compute the horizontal distribution H_{Y_r} associated to the connection α_{Y_r}. Let $z = [g, \rho, v] \in Y_r$ and $v_z = T_{(g,\rho,v)}\pi(T_eL_g(\eta), \alpha, u) \in T_zY_r$, with $\pi : G \times \mathfrak{m}_r^* \times V_r \to G \times_{G_m} (\mathfrak{m}_r^* \times V_r)$ the projection. Notice first that for any $\xi \in \mathfrak{g}$

$$\xi_{Y_r}([g, \rho, v]) = \left.\frac{d}{dt}\right|_{t=0} [\exp t\xi g, \rho, v] = T_{(g,\rho,v)}\pi(T_eL_g(\mathrm{Ad}_{g^{-1}}\xi), 0, 0).$$

Hence, for any $\nu \in \mathfrak{g}^*$, $(v_z, \nu) \in H_{Y_r}(z, \mu)$ if and only if

$$\omega_{Y_r}(z)(T_{(g,\rho,v)}\pi(T_eL_g(\mathrm{Ad}_{g^{-1}}\xi), 0, 0), v_z) = \langle \nu, \xi \rangle \quad \text{for any} \quad \xi \in \mathfrak{g}.$$

Using (3.5) we can rewrite this expression as

$$\langle \alpha + T_v\mathbf{J}_V(u), \mathrm{Ad}_{g^{-1}}\xi \rangle + \langle \rho + \mathbf{J}_V(v), [\mathrm{Ad}_{g^{-1}}\xi, \eta] \rangle + \Psi(m)(\mathrm{Ad}_{g^{-1}}\xi, \eta) = \langle \nu, \xi \rangle.$$

Since in this expression $\xi \in \mathfrak{g}$ is arbitrary it follows that

$$\nu = \mathrm{Ad}^*_{g^{-1}}\left(\alpha + T_v\mathbf{J}_V(u) - \mathrm{ad}^*_\eta(\rho + \mathbf{J}_V(v)) - \Psi(m)(\eta, \cdot)\right).$$

Consequently, we have

$$\begin{aligned} H_{Y_r}(z, \mu) = \big\{ (T_{(g,\rho,v)}\pi(T_eL_g(\eta), \alpha, u),\ \mathrm{Ad}^*_{g^{-1}}\left(\alpha + T_v\mathbf{J}_V(u) - \mathrm{ad}^*_\eta(\rho + \mathbf{J}_V(v)) - \Psi(m)(\eta, \cdot)\right)) \\ \mid \eta \in \mathfrak{g}, u \in v, \alpha \in \mathfrak{m}^* \big\}. \end{aligned} \tag{6.8}$$

We emphasize that this expression is well defined because if $(T_eL_g(\zeta), \mathrm{ad}^*_\zeta\rho, -\zeta \cdot v) \in \ker T_{(g,\rho,v)}\pi$, with $\zeta \in \mathfrak{g}_m$ then the second component in (6.8) is zero. Indeed,

$$\mathrm{Ad}^*_{g^{-1}}\left(\mathrm{ad}^*_\zeta\rho - T_v\mathbf{J}_V(\zeta \cdot v) - \mathrm{ad}^*_\zeta(\rho + \mathbf{J}_V(v)) - \Psi(m)(\zeta, \cdot)\right) = \mathrm{Ad}^*_{g^{-1}}\left(-T_v\mathbf{J}_V(\zeta \cdot v) - \mathrm{ad}^*_\zeta(\mathbf{J}_V(v))\right)$$

since $\Psi(m)(\zeta, \cdot) = 0$ because $\zeta \in \mathfrak{g}_m$. Now, since for any $\eta \in \mathfrak{g}_m$ we have $\langle T_v\mathbf{J}_V(\zeta \cdot v), \eta \rangle = -\omega_V(\eta \cdot v, \zeta \cdot v)$ and $-\langle \mathrm{ad}^*_\zeta(\mathbf{J}_V(v)), \eta \rangle = -\langle \mathbf{J}_V(v), [\zeta, \eta] \rangle$ we get

$$\begin{aligned} \langle -T_v\mathbf{J}_V(\zeta \cdot v) - \mathrm{ad}^*_\zeta(\mathbf{J}_V(v)), \eta \rangle &= \omega_V(\zeta \cdot v, \eta \cdot v) - \frac{1}{2}\omega_V([\zeta, \eta] \cdot v, v) \\ &= \omega_V(\zeta \cdot v, \eta \cdot v) - \frac{1}{2}\omega_V(\zeta \cdot (\eta \cdot v) - \eta \cdot (\zeta \cdot v), v) \\ &= \omega_V(\zeta \cdot v, \eta \cdot v) + \frac{1}{2}\omega_V(\eta \cdot v, \zeta \cdot v) - \frac{1}{2}\omega_V(\zeta \cdot v, \eta \cdot v) = 0. \end{aligned}$$

Lemma 6.2 *Let $\widetilde{G/G_m}$ be the holonomy bundle associated to the connection α_{G/G_m} that contains the point $(G_m, \mu) \in G/G_m \times \mathfrak{g}^*$. Then the holonomy bundle $\widetilde{Y_r}$ associated to the connection α_{Y_r} and containing $([e, 0, 0], \mu)$ is given by*

$$\widetilde{Y_r} = \{([g, \rho, v], \mathrm{Ad}^*_{g^{-1}}(\rho + \mathbf{J}_V(v)) + \nu) \mid [g, \rho, v] \in Y_r, (gG_m, \nu) \in \widetilde{G/G_m}\}. \tag{6.9}$$

Proof. We start by checking that any point of the form $\widetilde{y} = ([g, \rho, v], \mathrm{Ad}^*_{g^{-1}}(\rho + \mathbf{J}_V(v)) + \nu) \in Y_r \times \mathfrak{g}^*$, that satisfies $(gG_m, \nu) \in \widetilde{G/G_m}$, can be joined to $([e, 0, 0], \mu)$ via a horizontal curve for the connection α_{Y_r}. This will prove that the right hand side of (6.9) is included in its left hand side. Let $\gamma(t) = ([g](t), \nu(t))$ be a piecewise smooth horizontal curve for the connection α_{G/G_m} joining (G_m, μ) to (gG_m, ν). Using local sections for the projection $G \to G/G_m$, $\gamma(t)$ can be written as a finite concatenation of smooth curves of the form $(g_i(t)G_m, \nu(t))$, $i \in \{1, \ldots, n\}$ such that each $g_i : [0, 1] \to G$ is smooth. For the sake of simplicity we shall work with only one such curve. The claim follows by noticing that $y(t) := ([g(t), t\rho, tv], \mathrm{Ad}^*_{g(t)^{-1}}(t\rho + \mathbf{J}_V(tv)) + \nu(t))$ is a horizontal curve for α_{Y_r}.

In order to prove the converse inclusion we take an arbitrary element $([g, \rho, v], \nu) \in \widetilde{Y_r}$. By definition of the holonomy bundle we can construct piecewise smooth curves $g : [0, 1] \to G$, $\rho : [0, 1] \to \mathfrak{m}^*_r$, $v : [0, 1] \to V_r$, and $\nu : [0, 1] \to \mathfrak{g}^*$ such that $([g(t), \rho(t), v(t)], \nu(t))$ is a horizontal curve joining $([g, \rho, v], \nu)$ with $([e, 0, 0], \mu)$. Horizontality implies that

$$\begin{aligned} \dot{\nu}(t) &= \mathrm{Ad}^*_{g(t)^{-1}}(\dot{\rho}(t) + T_{v(t)}\mathbf{J}_V(\dot{v}(t)) - \mathrm{ad}^*_{g(t)^{-1}\dot{g}(t)}(\rho(t) + \mathbf{J}_V(v(t))) - \Psi(m)(g(t)^{-1}\dot{g}(t), \cdot)) \\ &= \frac{d}{dt}\left(\mathrm{Ad}^*_{g(t)^{-1}}(\rho(t) + \mathbf{J}_V(v(t)))\right) - \mathrm{Ad}^*_{g(t)^{-1}}\left(\Psi(m)(g(t)^{-1}\dot{g}(t), \cdot)\right). \end{aligned} \tag{6.10}$$

Let now $\widetilde{g}(t) = (g(t)G_m, \nu(t)) \in G/G_m \times \mathfrak{g}^*$ be the horizontal lift through (G_m, μ) of the curve $g(t)G_m \in G/G_m$ with respect to the connection α_{G/G_m}. This means that

$$\dot{\nu}(t) = -\mathrm{Ad}^*_{g(t)^{-1}}(\Psi(m)(g(t)^{-1}\dot{g}(t), \cdot)),$$

which allows us to rewrite (6.10) as

$$\dot{\nu}(t) = \frac{d}{dt}\left(\mathrm{Ad}^*_{g(t)^{-1}}(\rho(t) + \mathbf{J}_V(v(t))) + \nu(t)\right).$$

Consequently,

$$\nu(t) = \mathrm{Ad}^*_{g(t)^{-1}}(\rho(t) + \mathbf{J}_V(v(t))) + \nu(t)$$

and hence

$$([g, \rho, v], \nu) = ([g(1), \rho(1), v(1)], \nu(1)) = ([g(1), \rho(1), v(1)], \mathrm{Ad}^*_{g(1)^{-1}}(\rho(1) + \mathbf{J}_V(v(1)) + \nu(1))).$$

Since $(g(1)G_m, \nu(1)) \in \widetilde{G/G_m}$ we have that $([g, \rho, v], \nu)$ belongs to the right hand side of (6.9), as required. ▼

Lemma 6.3 *The holonomy groups $\mathcal{H}_{Y_r}$ and $\mathcal{H}_{G/G_m}$ corresponding to the connections α_{Y_r} and α_{G/G_m}, respectively, coincide, that is,*

$$\mathcal{H}_{Y_r} = \mathcal{H}_{G/G_m}.$$

The connection α_{G/G_m} is flat.

Proof. Take first $\nu \in \mathcal{H}_{G/G_m}$. By definition, there exists a piecewise smooth curve $g : [0, 1] \to G$ such that $g(0)G_m = g(1)G_m = G_m$ and whose horizontal lift $\widetilde{g(t)G_m} = (g(t)G_m, \nu(t))$ with respect to the connection α_{G/G_m} satisfies $\nu(1) - \nu(0) = \nu$. The horizontality of $\widetilde{g(t)G_m}$ implies that

$$\dot{\nu}(t) = -\mathrm{Ad}^*_{g(t)^{-1}}\left(\Psi(m)(g(t)^{-1}\dot{g}(t), \cdot)\right).$$

Let now $[g(t), 0, 0]$ be a loop on Y_r. The curve $([g(t), 0, 0], \nu(t))$ is its horizontal lift because its derivative lies on the right hand side of (6.8). In addition, this proves that $\nu = \nu(1) - \nu(0) \in \mathcal{H}_{Y_r}$.

In order to prove the converse inclusion take $\mu \in \mathcal{H}_{Y_r}$. Then there exists a piecewise smooth loop $y(t) = [g(t), \rho(t), v(t)]$ in Y_r such that $[g(0), \rho(0), v(0)] = [g(1), \rho(1), v(1)] = [e, 0, 0]$ and whose horizontal lift $\widetilde{y}(t) = ([g(t), \rho(t), v(t)], \mu(t))$ satisfies $\mu = \mu(1) - \mu(0)$. The functions $g : [0, 1] \to G$, $\rho : [0, 1] \to \mathfrak{m}_r^*$, and $v : [0, 1] \to V_r$ can be chosen, without loss of generality, so that $g(0) = g(1) = e$, $\rho(0) = \rho(1) = 0$, and $v(0) = v(1) = 0$. Let now $(g(t)G_m, \nu(t))$ be the horizontal lift of the loop $g(t)G_m$ with respect to the connection α_{G/G_m}, namely, the function $\nu(t)$ satisfies $\dot{\nu}(t) = -\mathrm{Ad}^*_{g(t)^{-1}}\left(\Psi(m)(g(t)^{-1}\dot{g}(t), \cdot)\right)$. Consequently,

$$\begin{aligned}\mu = \mu(1) - \mu(0) &= \int_0^1 \dot{\mu}(t)dt = \int_0^1 \left[\frac{d}{dt}\left(\mathrm{Ad}^*_{g(t)^{-1}}(\rho(t) + \mathbf{J}_V(v(t)))\right) - \mathrm{Ad}^*_{g(t)^{-1}}\left(\Psi(m)(g(t)^{-1}\dot{g}(t), \cdot)\right)\right] dt \\ &= \mathrm{Ad}^*_{g(1)^{-1}}(\rho(1) + \mathbf{J}_V(v(1))) - \mathrm{Ad}^*_{g(0)^{-1}}(\rho(0) + \mathbf{J}_V(v(0))) + (\nu(1) - \nu(0)) = \nu(1) - \nu(0) \in \mathcal{H}_{G/G_m}.\end{aligned}$$

As to the flatness of α_{G/G_m} notice that since $\mathrm{Lie}\left(\mathcal{H}_{G/G_m}\right) = \mathrm{Lie}\left(\mathcal{H}_{Y_r}\right) = \{0\}$ the tangent spaces to the holonomy bundles equal the horizontal distribution. Since the distribution associated to the holonomy bundles is integrable by general theory this shows that the horizontal distribution is integrable and hence the associated connection is flat. ▼

Suppose that $\mathbf{K}(m) = \pi_C(\mu)$ for some $\mu \in \mathfrak{g}^*$ and let $\widetilde{G/G_m}$ be the holonomy bundle associated to the connection α_{G/G_m} introduced in Proposition 6.1 that contains the point (G_m, μ). We will now show that for any $[g, \rho, v] \in Y_r$

$$\mathbf{K}(\phi[g, \rho, v]) = \pi_C(\mathrm{Ad}^*_{g^{-1}}(\rho + \mathbf{J}_V(v)) + \nu), \tag{6.11}$$

where $\nu \in \mathfrak{g}^*$ is any element such that $(gG_m, \nu) \in \widetilde{G/G_m}$. We start by showing that this expression is well defined. Let $[g, \rho, v], [g', \rho', v'] \in Y_r$ and $\nu, \nu' \in \mathfrak{g}^*$ be such that

$$[g, \rho, v] = [g', \rho', v'] \tag{6.12}$$

and

$$(gG_m, \nu), (g'G_m, \nu') \in \widetilde{G/G_m}. \tag{6.13}$$

The equality (6.12) implies that there exists an element $h \in G_m$ such that $g' = gh$, $\rho' = h^{-1} \cdot \rho$, and $v' = h^{-1} \cdot v$, while (6.13) guarantees that $(g'G_m, \nu') = (ghG_m, \nu') = (gG_m, \nu') \in \widetilde{G/G_m}$. Since $(gG_m, \nu) \in \widetilde{G/G_m}$ also, there exists an element $\tau \in \mathcal{H}_{G/G_m}$ such that $\nu' = \nu + \tau$. By the lemmas 6.1 and 6.3 we have $\mathcal{H}_{G/G_m} = \mathcal{H}_{Y_r} \subset \mathcal{H}$, which implies that $\pi_C(\tau) = 0$. Taking this into consideration as well as the G_m–equivariance of the momentum map $\mathbf{J}_V$, we conclude

$$\begin{aligned}\mathbf{K}(\phi([g', \rho', v'])) &= \pi_C \left(\mathrm{Ad}^*_{(gh)^{-1}} (h^{-1} \cdot \rho + \mathbf{J}_V(h^{-1} \cdot v)) + \nu' \right) = \pi_C \left(\mathrm{Ad}^*_{g^{-1}} (\rho + \mathbf{J}_V(v)) + \nu + \tau \right) \\ &= \pi_C \left(\mathrm{Ad}^*_{g^{-1}} (\rho + \mathbf{J}_V(v)) + \nu \right) = \mathbf{K}(\phi([g, \rho, v])),\end{aligned}$$

as required.

Let now $\widetilde{Y_r} \subset Y_r \times \mathfrak{g}^*$ and $\widetilde{M} \subset M \times \mathfrak{g}^*$ be the holonomy bundles associated to the connections α_{Y_r} and α and containing the points $([e, 0, 0], \mu)$ and (m, μ), respectively. We now show that

$$\text{if} \quad ([g, \rho, v], \zeta) \in \widetilde{Y_r} \quad \text{then} \quad (\phi([g, \rho, v]), \zeta) \in \widetilde{M}. \tag{6.14}$$

Indeed, if $([g, \rho, v], \zeta) \in \widetilde{Y_r}$ then there exists a piecewise smooth horizontal curve $\gamma_{Y_r}(t) = ([g(t), \rho(t), v(t)], \nu(t))$ in $Y_r \times \mathfrak{g}^*$ such that $\gamma_{Y_r}(0) = ([e, 0, 0], \mu)$ and $\gamma_{Y_r}(1) = ([g, \rho, v], \zeta)$. Since the map ϕ is a G–equivariant symplectomorphism, the curve $\gamma_M(t) := (\phi([g(t), \rho(t), v(t)]), \nu(t))$ is horizontal with respect to α. Indeed, $\gamma_M(t)$ is horizontal if and only if for any $\xi \in \mathfrak{g}$ we have

$$\omega(\phi(\lambda(t)))(\xi_M(\phi(\lambda(t))), T_{\lambda(t)}\phi \cdot \dot{\lambda}(t)) = \langle \dot{\nu}(t), \xi \rangle,$$

where $\lambda(t) = [g(t), \rho(t), v(t)]$. Since ϕ is G–equivariant this equality is equivalent to

$$\omega(\phi(\lambda(t)))(T_{\lambda(t)}\phi \cdot \xi_{Y_r}(\lambda(t)), T_{\lambda(t)}\phi \cdot \dot{\lambda}(t)) = \langle \dot{\nu}(t), \xi \rangle.$$

Given that ϕ is a symplectic map, this relation amounts to

$$\omega_{Y_r}(\lambda(t))(\xi_{Y_r}(\lambda(t)), \dot{\lambda}(t)) = \langle \dot{\nu}(t), \xi \rangle$$

which is a true relation by the horizontality of γ_{Y_r}. The statement in (6.14) implies that we can write

$$\mathbf{K}(\phi([g, \rho, v])) = \pi_C(\zeta) = \pi_C(\widetilde{\mathbf{K}_{Y_r}}([g, \rho, v], \zeta)),$$

where $\widetilde{\mathbf{K}_{Y_r}} : \widetilde{Y_r} \subset Y_r \times \mathfrak{g}^* \to \mathfrak{g}^*$ is the projection into the $\mathfrak{g}^*$ factor. Expression (6.9) guarantees that $\zeta = \widetilde{\mathbf{K}_{Y_r}}([g, \rho, v], \zeta) = \mathrm{Ad}^*_{g^{-1}}(\rho + \mathbf{J}_V(v)) + \nu$ with $(gG_m, \nu) \in \widetilde{G/G_m}$ and hence $\mathbf{K}(\phi[g, \rho, v]) = \pi_C(\mathrm{Ad}^*_{g^{-1}}(\rho + \mathbf{J}_V(v)) + \nu)$, as required.

In order to conclude the proof we will show that the expressions (3.7) and (6.11) coincide. This obviously amounts to showing that

$$\Theta_g(\mathbf{K}(m)) = \pi_C(\nu), \tag{6.15}$$

where we recall that $\nu \in \mathfrak{g}^*$ is any element such that $(gG_m, \nu) \in \widetilde{G/G_m}$, where $\widetilde{G/G_m}$ is the holonomy bundle associated to the connection α_{G/G_m} that contains the point (G_m, μ). By hypothesis, the group G is connected so we can assume the existence of a piecewise smooth curve $g(t) \subset G$ such that $g(0) = e$ and $g(1) = g$. Let $(g(t)G_m, \mu(t))$ be the horizontal lift of $g(t)G_m$ that satisfies the initial condition $(g(0)G_m, \mu(0)) = (G_m, \mu)$. By construction, the curve $\mu(t)$ satisfies the differential equation

$$\langle \dot{\mu}(t), \xi \rangle = -\langle \mathrm{Ad}^*_{g(t)^{-1}} \Psi(m) \left(T_{g(t)} L_{g(t)^{-1}} \cdot \dot{g}(t), \cdot \right), \xi \rangle = -\Psi(m) \left(T_{g(t)} L_{g(t)^{-1}} \cdot \dot{g}(t), \mathrm{Ad}_{g(t)^{-1}} \xi \right), \tag{6.16}$$

for any $\xi \in \mathfrak{g}$ and, additionally, $\mu(1) = \nu + \lambda$, for some $\lambda \in \mathcal{H}_{G/G_m}$.

We now study the left hand side of (6.15):

$$\Theta_g(\mathbf{K}(m)) = \mathbf{K}(g \cdot m) = \pi_C \left(\widetilde{\mathbf{K}}(g \cdot m, \tau) \right), \tag{6.17}$$

for some $\tau \in \mathfrak{g}^*$ such that $(g \cdot m, \tau) \in \widetilde{M}$, where $\widetilde{M}$ is the holonomy bundle used in the construction of the cylinder valued momentum map $\mathbf{K}$ which contains the point (m, μ). Let $g(t)$ be the same curve in G that we considered in the previous paragraph. The curve $g(t) \cdot m$ in M clearly links the points m and $g \cdot m$. Let $(g(t) \cdot m, \tau(t)) \subset \widetilde{M}$ be the horizontal lift of $g(t) \cdot m$ with initial condition $(g(0) \cdot m, \tau(0)) = (m, \mu)$. The curve $\tau(t)$ satisfies the differential equation

$$\langle \dot{\tau}(t), \xi \rangle = \omega(g(t) \cdot m) \left(\xi_M(g(t) \cdot m), \frac{d}{dt} (g(t) \cdot m) \right),$$

for any $\xi \in \mathfrak{g}$. Since $\frac{d}{dt}(g(t) \cdot m) = T_m \Phi_{g(t)} \cdot \left(T_{g(t)} L_{g(t)^{-1}} \cdot \dot{g}(t) \right)_M (m)$ we can rewrite

$$\begin{aligned} \langle \dot{\tau}(t), \xi \rangle &= \omega(g(t) \cdot m) \left(T_m \Phi_{g(t)} \cdot \left(\mathrm{Ad}_{g(t)^{-1}} \xi \right)_M (m), T_m \Phi_{g(t)} \cdot \left(T_{g(t)} L_{g(t)^{-1}} \cdot \dot{g}(t) \right)_M (m) \right) \\ &= \omega(m) \left(\left(\mathrm{Ad}_{g(t)^{-1}} \xi \right)_M (m), \left(T_{g(t)} L_{g(t)^{-1}} \cdot \dot{g}(t) \right)_M (m) \right) = \Psi(m) \left(\mathrm{Ad}_{g(t)^{-1}} \xi, T_{g(t)} L_{g(t)^{-1}} \cdot \dot{g}(t) \right), \end{aligned}$$

which coincides with (6.16) and hence shows that $\mu(t) = \tau(t)$ since $\mu = \mu(0) = \tau(0)$. Therefore $\tau(1) = \nu + \lambda = \tau$. If we insert this in (6.17) we obtain

$$\Theta_g(\mathbf{K}(m)) = \pi_C \left(\widetilde{\mathbf{K}}(g \cdot m, \tau) \right) = \pi_C(\tau) = \pi_C(\nu + \lambda).$$

Since $\mathcal{H}_{Y_r} \subset \mathcal{H}$, by Lemma 6.1, and $\mathcal{H}_{Y_r} = \mathcal{H}_{G/G_m}$, by Lemma 6.3, we have that $\mathcal{H}_{G/G_m} \subset \mathcal{H}$. Thus, since $\lambda \in \mathcal{H}_{G/G_m}$, we have

$$\Theta_g(\mathbf{K}(m)) = \pi_C(\nu + \lambda) = \pi_C(\nu),$$

as required. ■

6.3 Proof of Proposition 5.1

We start by noticing that the right hand side of (5.2) makes sense due to the closedness hypothesis on $\mathcal{H}$. Indeed, this condition and the fact that $\mathcal{H}$ is zero dimensional imply that $\mathfrak{g}^*/\mathcal{H}$, and therefore $\mathfrak{g}/\mathcal{T}$, are Abelian Lie groups whose Lie algebras can be naturally identified with $\mathfrak{g}^*$ and $\mathfrak{g}$, respectively. This identification is used in (5.2), where we think of $T_m(L_{\mathbf{J}(m)^{-1}} \circ \mathbf{J})(v_m) \in \mathrm{Lie}(\mathfrak{g}/\mathcal{T})$ as an element of $\mathfrak{g}$.

In the sequel we will use the following notation: given $\mu \in \mathfrak{g}^*$ arbitrary, we denote by $\xi_\mu \in \mathfrak{g}$ the unique element such that $\mu = (\xi_\mu, \cdot)$.

We now compute $T_m \mathbf{J}(v_m)$. Let $\mu + \mathcal{H} := \mathbf{K}(m)$ and hence $\mathbf{J}(m) = \xi_\mu + \mathcal{T}$. We have

$$T_m \mathbf{J}(v_m) = T_m(\bar{f}^{-1} \circ \mathbf{K})(v_m) = T_{\mu + \mathcal{H}} \bar{f}^{-1}(T_m \mathbf{K}(v_m)) = T_{\mu + \mathcal{H}} \bar{f}^{-1}(T_\mu \pi_C(\nu)),$$

where the element $\nu \in \mathfrak{g}^*$ is given by

$$\langle \nu, \eta \rangle = \mathbf{i}_{\eta_M} \omega(m)(v_m), \quad \text{for all} \quad \eta \in \mathfrak{g}. \tag{6.18}$$

Since $(\bar{f}^{-1} \circ \pi_C)(\rho) = \xi_\rho + \mathcal{T}$ for any $\rho \in \mathfrak{g}^*$, we can write

$$T_{\mu + \mathcal{H}} \bar{f}^{-1}(T_\mu \pi_C(\nu)) = T_\mu \left(\bar{f}^{-1} \circ \pi_C \right)(\nu) = \left. \frac{d}{dt} \right|_{t=0} \left(\bar{f}^{-1} \circ \pi_C \right)(\mu + t\nu) = \left. \frac{d}{dt} \right|_{t=0} (\xi_\mu + t\xi_\nu + \mathcal{T}).$$

Hence,

$$T_m \mathbf{J}(v_m) = \left. \frac{d}{dt} \right|_{t=0} (\xi_\mu + t\xi_\nu + \mathcal{T}) \in T_{\xi_\mu + \mathcal{T}} (\mathfrak{g}/\mathcal{T}).$$

Now,

$$\begin{aligned} \left(T_m(L_{\mathbf{J}(m)^{-1}} \circ \mathbf{J})(v_m), \xi \right) &= \left(T_{\mathbf{J}(m)} L_{\mathbf{J}(m)^{-1}}(T_m \mathbf{J}(v_m)), \xi \right) \\ &= \left(\left. \frac{d}{dt} \right|_{t=0} (-\xi_\mu + \mathcal{T}) + (\xi_\mu + t\xi_\nu + \mathcal{T}), \xi \right) = (\xi_\nu, \xi) = \langle \nu, \xi \rangle = \mathbf{i}_{\xi_M} \omega(m)(v_m), \end{aligned}$$

where the last equality is a consequence of (6.18). ■

6.4 Proof of Theorem 5.1

We start by assuming that the $\mathfrak{g}$–action on (M, ω) has an associated G–valued momentum map $\mathbf{A} : M \to G$ and we will show that $\mathcal{H} \subset f(\ker \exp)$.

Let $\mu \in \mathcal{H}$. The definition of the holonomy group $\mathcal{H}$ implies the existence of a piecewise smooth loop $m : [0,1] \to M$ at the point m, that is, $m(0) = m(1) = m \in M$, whose horizontal lift $\widetilde{m}(t) = (m(t), \mu(t))$ starting at the point $(m, 0)$ satisfies $\mu = \mu(1)$. The horizontality of $\widetilde{m}(t)$ implies that

$$\langle \dot{\mu}(t), \xi \rangle = \mathbf{i}_{\xi_M} \omega(m(t))(\dot{m}(t)) = \left(T_{m(t)} \left(L_{\mathbf{A}(m(t))^{-1}} \circ \mathbf{A} \right) (\dot{m}(t)), \xi \right),$$

for any $\xi \in \mathfrak{g}$ or, equivalently,

$$\dot{\mu}(t) = f \left(\left. \frac{d}{ds} \right|_{s=0} \mathbf{A}(m(t))^{-1} \mathbf{A}(m(s)) \right). \tag{6.19}$$

Fix $t_0 \in [0,1]$. Since the exponential map $\exp : \mathfrak{g} \to G$ is a local diffeomorphism, there exists a smooth curve $\xi : I_{t_0} := (t_0 - \epsilon, t_0 + \epsilon) \to \mathfrak{g}$, for $\epsilon > 0$ sufficiently small, such that for any $s \in (-\epsilon, \epsilon)$

$$\mathbf{A}(m(t_0 + s)) = \exp \xi(t_0 + s) \mathbf{A}(m(t_0)). \tag{6.20}$$

We now reformulate locally the expression (6.19) using the function $\xi : I_{t_0} \to \mathfrak{g}$. Let $\tau, \sigma \in (-\epsilon, \epsilon)$ be such that $t = t_0 + \tau$ and $s = t_0 + \sigma$. Expression (6.19) can be rewritten in I_{t_0} as

$$\begin{aligned}
\frac{d}{d\tau} \mu(t_0 + \tau) &= f \left(\left. \frac{d}{d\sigma} \right|_{\sigma=\tau} \mathbf{A}(m(t_0 + \tau))^{-1} \mathbf{A}(m(t_0 + \sigma)) \right) \\
&= f \left(\left. \frac{d}{d\sigma} \right|_{\sigma=\tau} \exp(-\xi(t_0 + \tau)) \exp \xi(t_0 + \sigma) \mathbf{A}(m(t_0))^{-1} \mathbf{A}(m(t_0)) \right) \\
&= f \left(\left. \frac{d}{d\sigma} \right|_{\sigma=\tau} \exp \left(\xi(t_0 + \sigma) - \xi(t_0 + \tau) \right) \right) = f \left(T_0 \exp \left(\left. \frac{d}{d\sigma} \right|_{\sigma=\tau} \left(\xi(t_0 + \sigma) - \xi(t_0 + \tau) \right) \right) \right) \\
&= f \left(\left. \frac{d}{d\sigma} \right|_{\sigma=\tau} \left(\xi(t_0 + \sigma) - \xi(t_0 + \tau) \right) \right) = f \left(\left. \frac{d}{d\sigma} \right|_{\sigma=\tau} \xi(t_0 + \sigma) \right) = f \left(\frac{d}{d\tau} \xi(t_0 + \tau) \right),
\end{aligned}$$

which shows that for any $t \in I_{t_0}$

$$\dot{\mu}(t) = f(\dot{\xi}(t)). \tag{6.21}$$

We now cover the interval $[0,1]$ with a finite number of intervals $I_1, \ldots, I_n$ such that in each of them we define a function $\xi_i : I_i \to \mathfrak{g}$ that satisfies (6.20) and (6.21). We now write $I_i = [a_i, a_{i+1}]$, with $i \in \{1, \ldots, n\}$, $a_1 = 0$, and $a_{n+1} = 1$. Using these intervals, since $\mu(0) = 0$, we can write

$$\begin{aligned}
\mu = \mu(1) &= \int_0^1 \dot{\mu}(t) dt = \int_{I_1} \dot{\mu}(t) dt + \cdots + \int_{I_n} \dot{\mu}(t) dt \\
&= f \left(\int_{I_1} \dot{\xi}_1(t) dt + \cdots + \int_{I_n} \dot{\xi}_n(t) dt \right) = f \left(\xi_1(a_2) - \xi_1(a_1) + \cdots + \xi_n(a_{n+1}) - \xi_n(a_n) \right).
\end{aligned} \tag{6.22}$$

The construction of the intervals I_i, $i \in \{1, \ldots, n\}$, implies that $\mathbf{A}(m(a_i)) = \exp \xi_i(a_i) \mathbf{A}(m(a_i))$. Hence

$$\exp \xi_i(a_i) = e \tag{6.23}$$

and hence $\xi_i(a_i) \in \ker \exp$ for all $i \in \{1, \ldots, n\}$. We also have that

$$\begin{aligned}
\mathbf{A}(m(1)) = \mathbf{A}(m(a_{n+1})) &= \exp \xi_n(a_{n+1}) \mathbf{A}(m(a_n)) = \exp \xi_n(a_{n+1}) \exp \xi_{n-1}(a_n) \mathbf{A}(m(a_{n-1})) \\
&= \exp \xi_n(a_{n+1}) \exp \xi_{n-1}(a_n) \cdots \exp \xi_1(a_2) \mathbf{A}(m(a_1)) = \exp(\xi_1(a_2) + \cdots + \xi_n(a_{n+1})) \mathbf{A}(m(0)).
\end{aligned}$$

Since $m(0) = m(1) = m$ we have $\mathbf{A}(m(0)) = \mathbf{A}(m(1))$ and, consequently, $\exp(\xi_1(a_2) + \cdots + \xi_n(a_{n+1})) = e$, which implies that $\xi_1(a_2) + \cdots + \xi_n(a_{n+1}) \in \ker \exp$. If we substitute this relation and (6.23) in (6.22) we obtain that $\mu \in f(\ker \exp)$, which proves the inclusion $\mathcal{H} \subset f(\ker \exp)$.

We now show that $\mathcal{H}$ is closed in $\mathfrak{g}^*$. The inclusion $\mathcal{H} \subset f(\ker \exp)$, the closedness of $\ker \exp$ in $\mathfrak{g}$, and the fact that f is an isomorphism imply that

$$\overline{\mathcal{H}} \subset \overline{f(\ker \exp)} = f(\ker \exp).$$

Because G is Abelian, $\ker \exp$ is a zero dimensional subgroup of $(\mathfrak{g}, +)$ and hence $\overline{\mathcal{H}}$ is a zero dimensional subgroup of $\mathfrak{g}^*$. This implies that $\overline{\mathcal{H}} \subset \mathcal{H}$. Indeed, for any element $\mu \in \overline{\mathcal{H}}$ there exists an open neighborhood $U_\mu \subset \mathfrak{g}^*$ such that $U_\mu \cap \overline{\mathcal{H}} = \{\mu\}$ ($\overline{\mathcal{H}}$ is zero dimensional). As $\mu \in \overline{\mathcal{H}}$ we have that $\emptyset \neq U_\mu \cap \mathcal{H} \subset U_\mu \cap \overline{\mathcal{H}} = \{\mu\}$, which implies that $\mu \in \mathcal{H}$. This shows that $\mathcal{H} = \overline{\mathcal{H}}$ and therefore that $\mathcal{H}$ is closed in $\mathfrak{g}^*$.

Assume now that $f(\ker \exp) \subset \mathcal{H}$. The hypothesis on the existence of a Lie group valued momentum map implies, via the inclusion (5.3) that we just proved, that $f(\ker \exp) = \mathcal{H}$ and that $\mathcal{H}$ is closed in $\mathfrak{g}^*$. Proposition 5.1 implies that $\mathbf{J} : M \to \mathfrak{g}/\ker \exp \simeq G$ is a G–valued momentum map for the $\mathfrak{g}$–action on (M, ω). We now show that $\mathbf{A}$ and $\mathbf{J}$ differ by a constant in G. The expression (5.1) for $\mathbf{A}$ and (5.2) for $\mathbf{J}$ imply that for any $\xi \in \mathfrak{g}$ and any $v_m \in T_mM$ we have

$$\left(T_m(L_{\mathbf{A}(m)^{-1}} \circ \mathbf{A})(v_m), \xi\right) = \mathbf{i}_{\xi_M}\omega(m)(v_m) = \left(T_m(L_{\mathbf{J}(m)^{-1}} \circ \mathbf{J})(v_m), \xi\right),$$

which implies that $T\mathbf{J} = T\mathbf{A}$. Since the manifold M is connected we have that $\mathbf{A}$ and $\mathbf{J}$ coincide up to a constant element in $\mathbf{G}$.

The last claim in the theorem is a straightforward corollary of Proposition 5.1. ■

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